

LAGRANGIAN COBORDISM IN LEFSCHETZ FIBRATIONS.

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ABSTRACT. Given a symplectic manifold (M^{2n}, ω) we study Lagrangian cobordisms $V \subset E$ where E is the total space of a Lefschetz fibration having M as generic fiber. We prove a generation result for these cobordisms in the appropriate derived Fukaya category. As a corollary, we analyze the relations among the Lagrangian submanifolds $L \subset M$ that are induced by these cobordisms. This leads to a unified treatment - and a generalization - of the two types of relations among Lagrangian submanifolds of M that were previously identified in the literature: those associated to Dehn twists that were discovered by Seidel [Sei2] and the relations induced by cobordisms in trivial symplectic fibrations described in our previous work [BC2, BC3].

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1. INTRODUCTION

1.1. Motivation. The derived Fukaya category $D\mathcal{Fuk}(N)$ of a symplectic manifold (N, ω) is a triangulated category whose objects are obtained as the completion of a certain class - here denoted by $\mathcal{L}(N)$ - of Lagrangian submanifolds of N . The completion can be summarized as follows. As a set, each Lagrangian L can be described as a collection of sets each consisting of intersection points $L' \cap L$ where L' is a variable Lagrangian transverse to L . This family of intersection points can be assembled in a family of vector spaces $\mathbb{Z}_2\langle L' \cap L \rangle$ again with L' viewed as a variable. In the absence of some coherence relations among all these vector spaces this is obviously not a useful description of L . However, given some almost complex structure J , compatible with ω , there are natural relations among the vector spaces $\mathbb{Z}_2\langle - \cap L \rangle$ that reflect the existence of J -holomorphic curves with Lagrangian boundary conditions along families $L_1, \dots, L_k \in \mathcal{L}(N)$ and L . The formal way to express this is to construct first an A_∞ -category $\mathcal{Fuk}(N)$ called the Fukaya category of N with objects $\mathcal{L}(N)$, with morphisms the vector spaces $\text{hom}(L', L'') = \mathbb{Z}_2\langle L' \cap L'' \rangle$ and so that the higher multiplications μ_k are given by counts of J -holomorphic polygons with boundary components along L_1, L_2, \dots, L_{k+1} . In this formalism the family $\mathbb{Z}_2\langle - \cap L \rangle$ becomes a module over $\mathcal{Fuk}(N)$, called the Yoneda module associated to L , $\mathcal{Y}(L)$. The modules over an A_∞ -category are algebraic objects that behave in ways very similar to chain complexes. In particular, given a morphism between two modules $f : \mathcal{M} \rightarrow \mathcal{M}'$, one can take the cone over it $\mathcal{M}'' = \text{cone}(f)$, which is a module given by a

formula similar to the cone over a chain map. The category $D\mathcal{Fuk}(N)$ has as objects all the modules that can be obtained by iterated cones from the Yoneda modules. The morphisms in this category are the homology classes of the module morphisms. The exact triangles are the homology images of the chain-level triangles of morphisms that are quasi-isomorphic to the module-level cone attachments. We refer to [Sei3, Section 3e] for the detailed construction. We remark that our variant of the derived Fukaya category is not completed with respect to idempotents, by contrast to other versions of this notion that are present in the literature. Note also that in this paper we work with ungraded A_∞ -categories, in particular there are no shift operations.

Two closely inter-related types of results are key from this perspective. The first is decomposition results, that show that all objects in some class can be decomposed in $D\mathcal{Fuk}(-)$ in terms of basic objects, similarly to the way a CW -complex can be decomposed into cells. The second one is constructive results producing exact triangles in $D\mathcal{Fuk}(-)$ out of geometric structures or operations.

1.2. Main result. The main aim of this paper is to prove a decomposition result for a class of Lagrangian submanifolds with cylindrical ends - called cobordisms - that are embedded in the total space of a Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$. We consider here such cobordisms V with “negative” ends only: outside of a compact subset, the projection of V to \mathbb{C} is a union of rays of the type $\ell_i = (-\infty, a_i] \times \{i\}$, $i \in \mathbb{N}$. Such cobordisms will be called negatively-ended.

We work with uniformly monotone Lagrangians and with a class of Lefschetz fibrations that satisfy a strong variant of the monotonicity condition - see §3.1, §3.2 for the definitions. Let $\mathcal{L}^*(E)$ be the class of these cobordisms in E . The superscript $-^*$ will denote at all times below the monotonicity constraint imposed on the Lagrangians involved. We denote by \mathcal{A} the universal Novikov ring over the base field \mathbb{Z}_2 . The Fukaya categories in this paper will generally be over the field \mathcal{A} . Finally, recall that we work at all times in an ungraded context.

We state here the main decomposition result and refer to §4.1 where the result is restated after making the various ingredients more precise. Our conventions and notation regarding iterated cone decompositions are explained in §3.1.1. Henceforth we make the following standing assumption: all our Lefschetz fibrations E are assumed to have a positive dimensional fiber (hence $\dim_{\mathbb{R}} E \geq 4$).

Theorem A. *There exists a Fukaya category with objects the cobordisms in $\mathcal{L}^*(E)$. Let $D\mathcal{Fuk}^*(E)$ be the associated derived Fukaya category. Consider one object, $V \in \mathcal{L}^*(E)$, fix points $z_i \in \ell_i$ along the rays associated to V and let $L_i = V \cap \pi^{-1}(z_i)$. Let T_i be the thimbles associated to the curves t_i as in Figure 1, and let $\gamma_i L_i \subset E$ be obtained by the (union of) parallel transports of L_i along the curve γ_i , in the same figure.*

There exist finite rank \mathcal{A} -modules E_k , $1 \leq k \leq m$, and an iterated cone decomposition taking place in $D\mathcal{Fuk}^*(E)$:

$$V \cong (T_1 \otimes E_1 \rightarrow T_2 \otimes E_2 \rightarrow \dots \rightarrow T_m \otimes E_m \rightarrow \gamma_s L_s \rightarrow \gamma_{s-1} L_{s-1} \rightarrow \dots \rightarrow \gamma_2 L_2) .$$

The precise meaning of the notation in the last formula will be explained in §3.1.1. The \mathcal{A} -modules E_i are made explicit in the proof - see (57). For the time being, let us only mention that they are obtained as Floer homologies between V and certain Lagrangian spheres constructed in an auxiliary Lefschetz fibration associated to E .

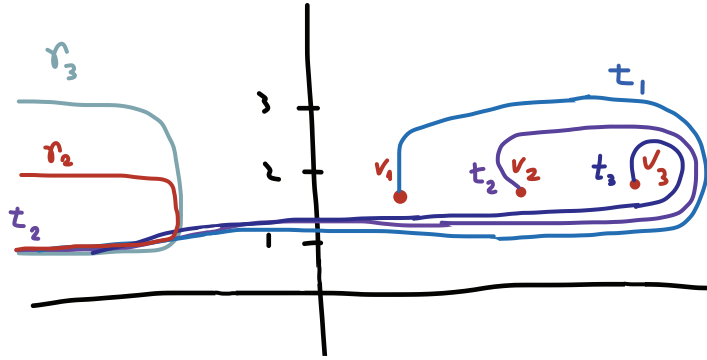


FIGURE 1. The curves γ_i , and the curves t_j emanating from the critical values v_j of the Lefschetz fibration.

1.3. Some consequences. Cobordisms are of interest not only for their own sake but also because they can be viewed as relators among their ends, in the sense of the usual cobordism relation. In this direction, one of the main consequences of Theorem A is that each such cobordism V produces an iterated cone decomposition inside $D\mathcal{Fuk}^*(M)$, where $M = \pi^{-1}(z_1)$ is the general fiber of E . This cone decomposition expresses the end L_1 of V as an iterated cone involving the ends L_i , $i \geq 2$ and the vanishing cycles of the singularities of π - see §5.1. Thus, cobordisms in E and the triangular decompositions in the (derived) Fukaya category of the fiber are intimately related - see Corollary 5.1.1.

To discuss a further consequence, recall that to any triangulated category \mathcal{C} one can associate a Grothendieck group $K_0\mathcal{C}$ defined as the quotient of the free abelian group generated by the objects of \mathcal{C} modulo the relations $B = A + C$ associated to each exact triangle $A \rightarrow B \rightarrow C$. We remark that in this paper we work with ungraded categories, hence our Grothendieck groups will always be 2-torsion (i.e. $2A = 0$ for every $A \in K_0\mathcal{C}$).

Another application of Theorem A - see §5.2 - is to give a description of the Grothendieck group $K_0D\mathcal{Fuk}^*(M)$ as an “algebraic” cobordism group. To explain this result we focus here on the case of the trivial fibration $E = \mathbb{C} \times M$ even if we establish the relevant results in more

generality in the paper. Recall from [BC3] the definition of the cobordism group $\Omega_{Lag}^*(M)$. It is the quotient of the free abelian group generated by the objects in $\mathcal{L}^*(M)$ modulo the relations $L_1 + L_2 + \dots + L_s = 0$ for each negatively-ended cobordism $V \subset \mathbb{C} \times M$ whose ends are L_1, \dots, L_s . For every $i \in \mathbb{N}$ there is a natural restriction operation that associates to a cobordism V its i -th end. These operations admit extensions to all objects of $D\mathcal{Fuk}^*(\mathbb{C} \times M)$. The i -th end of an object \mathcal{M} in $D\mathcal{Fuk}^*(\mathbb{C} \times M)$ is denoted by $[\mathcal{M}]_i \in \mathcal{Ob}(D\mathcal{Fuk}^*(M))$. It is natural to define an algebraic cobordism group $\Omega_{Alg}^*(M)$ as the free abelian group generated by the (isomorphism classes of) objects of $D\mathcal{Fuk}^*(M)$ modulo the relations $\sum_i [\mathcal{M}]_i = 0$ for each object \mathcal{M} of $D\mathcal{Fuk}^*(\mathbb{C} \times M)$. Equivalently, $\Omega_{Alg}^*(M)$ is defined in a similar way to $\Omega_{Lag}^*(M)$ only that the generators and relations now come also from the non-geometric objects in $D\mathcal{Fuk}^*(M)$ and $D\mathcal{Fuk}^*(\mathbb{C} \times M)$. There is an obvious map $q : \Omega_{Lag}^*(M) \rightarrow \Omega_{Alg}^*(M)$. A consequence of Theorem A, Corollary 5.2.3, is that there exists a group isomorphism

$$\Theta_{Alg} : \Omega_{Alg}^*(M) \rightarrow K_0 D\mathcal{Fuk}^*(M)$$

so that the composition $\Theta_{Alg} \circ q$ coincides with the Lagrangian Thom morphism

$$(1) \quad \Theta : \Omega_{Lag}^*(M) \rightarrow K_0 D\mathcal{Fuk}^*(M)$$

previously introduced in [BC3]. One of the reasons why this is of interest is that this result should shed some light on the kernel of Θ which is at present somewhat mysterious. Another implication of the fact that Θ_{Alg} is an isomorphism appears in Corollary 5.2.4 which asserts that the obvious map $\Omega_{Lag}^*(M) \rightarrow QH_*(M)$ admits an extension to $\Omega_{Alg}^*(M)$. Here $QH_*(M)$ stands for the quantum homology of the ambient manifold M .

Finally, we also obtain a periodicity result for K_0 - Corollary 5.2.6:

$$(2) \quad K_0(D\mathcal{Fuk}^*(\mathbb{C} \times M)) \cong \mathbb{Z}_2[t] \otimes K_0(D\mathcal{Fuk}^*(M)) .$$

Here t is a formal variable whose role will become clear in the proof (roughly speaking, different powers of t are used to label the K_0 -classes associated to different ends of a cobordism, or more generally, “ends” of an object of $D\mathcal{Fuk}^*(\mathbb{C} \times M)$).

1.4. Relation to previous work. Theorem A can be viewed as a simultaneous generalization of the two previously known methods to produce exact triangles in the derived Fukaya category.

The first such method is due to Seidel [Sei2], [Sei3, Chapter III, Section 17] and, in its basic form, it associates an exact triangle of the form:

$$(3) \quad \tau_S L \rightarrow L \rightarrow S \otimes HF(S, L)$$

to the Dehn twist $\tau_S : M \rightarrow M$ corresponding to a Lagrangian sphere S and any $L \in \mathcal{L}^*(M)$ (Seidel works in an exact setting, but as we will see below, this triangle remains valid in the

monotone context too. Other cases have been treated in the literature too, e.g. see [Oh3] for the case of Lagrangians with vanishing Maslov class in Calabi-Yau manifolds). Seidel also considers a Fukaya category $\mathcal{Fuk}(\pi)$ associated to a Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$, [Sei3, Sei4]. In our setting, this category corresponds to the full and faithful subcategory of $\mathcal{Fuk}^*(E)$ generated by the thimbles T_i . He also proves a decomposition result for this category that, in our context, essentially implies the statement of Theorem A in the special case when V has a single end. This category is related to mirror symmetry questions [?] and, indeed, cobordisms with a single end appear in relation to mirror symmetry, see for instance [HAV]. Cobordisms with multiple ends as well as a category somewhat similar to $\mathcal{Fuk}^*(E)$ appear in the recent paper [AS].

The second method appears in our previous paper [BC3]. It is shown there that if $V \subset \mathbb{C} \times M$ is a cobordism, then the ends of V are related by a cone-decomposition in $D\mathcal{Fuk}^*(M)$. This decomposition coincides with the one in Corollary 5.1.1 below when E is the trivial fibration $\mathbb{C} \times M$. Nevertheless, we remark that the statement of Theorem A - which concerns decompositions of cobordisms - is new even for the trivial fibration.

The exact triangle associated to a Dehn twist and the exact triangle obtained through the cobordism machinery coincide when there is a single and transverse intersection between S and L . This can be shown by methods already in the literature. For example, this follows from a combination of the results from [Sei1] and [BC3] (see also [FOOO, Oh3] for an earlier approach). In this case, Seidel's exact triangle coincides with the surgery exact sequence which is associated to a specific cobordism (in $\mathbb{C} \times M$) whose ends are $\tau_S L, L, S$. This cobordism is constructed as the trace of the Lagrangian surgery at the intersection point $S \cap L$. Theorem A and its proof go beyond this case and further clarify the interplay between these two constructions.

From a technical standpoint, we rely heavily on Seidel's work [Sei3] - in particular, the detailed constructions of $D\mathcal{Fuk}(-)$, which we adapt to the monotone setting. We also build on Seidel's set-up of Lefschetz fibrations in the symplectic framework in [Sei3, Sei2]. There is also a variety of other specific points where our work is related to his and these are mentioned along the text. We also make heavy use of the constructions in our previous papers [BC2, BC3]. At the same time, in attempt to keep this text readable we will recall several ingredients from [BC2, BC3] that are crucial for the present paper.

1.5. Outline of the paper. Most of the paper is aimed towards the proof of Theorem A. This proof requires two preliminaries. The first is contained in §2. That section contains the general set-up and terminology concerning Lefschetz fibrations. We introduce a special type of such fibrations called *tame* which are basically Lefschetz fibrations over \mathbb{C} that are symplectically trivial *outside* a U -like region in the plane. (See Definition 2.2.2. See also

Figure 3 on page 13, where the complement of the U -like region is denoted by \mathcal{W} .) Tame fibrations are much easier to handle in the technical parts of the proof. One of the reasons is that cylindrical ends can be easily moved around in the trivial region since parallel transport is trivial over there. Additionally, the Fukaya A_∞ category with objects cobordisms in such fibrations can be defined following closely the constructions in [BC3]. In §2.3 we show that any Lefschetz fibration with a finite number of (simple) singularities can be transformed into a tame one. As a consequence, Theorem A follows from the corresponding result - stated as Theorem 4.2.1 - for tame fibrations.

The second preliminary is the construction of the Fukaya category $\mathcal{Fuk}^*(E)$. This is described in §3. We first give the main elements of the construction when the Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$ is tame. In this case, the construction that appears in [BC3] applies essentially without change and we review the main steps. We then indicate the modifications needed to define such a category in the general case. In the discussion below we will mainly assume that all critical values of the Lefschetz fibration $E \rightarrow \mathbb{C}$ lie in the upper half-plane. Moreover, the objects in our categories will be cobordisms in E whose projection to \mathbb{C} is contained in the upper half-plane and that are cylindrical outside some fixed strip $[-a, a] \times \mathbb{R}$. (See §3.3, §4.1 for the precise setting.)

With this preparation, the actual proof of Theorem A is contained in §4 and it consists of three main ingredients. The first one deals with decompositions of cobordisms V' - called remote with respect to E - that are included in the total space E' of a Lefschetz fibration that coincides with E over the upper half-plane. The defining property of such a V' is that it can be moved inside E' away from the critical points of $E \rightarrow \mathbb{C}$, so that its only intersection with an object X of $\mathcal{Fuk}^*(E)$ occurs in the region where both V' and X are cylindrical. We show in §4.3 that such a remote cobordism viewed as a module over $\mathcal{Fuk}^*(E)$ admits a decomposition just as the one in the statement of Theorem A but without any of terms $T_i \otimes E_i$. The second step, in §4.4, shows how to transform a general cobordism V into a remote one. This is a geometric step, potentially of independent interest. It is done, roughly speaking, by placing V inside a new Lefschetz fibration E' obtained from E by adding singularities over the lower half-plane and showing that the cobordism $V' \subset E'$ obtained as an iterated Dehn twist of V , $V' = (\tau_{S_m} \circ \dots \circ \tau_{S_i} \circ \dots \circ \tau_{S_1})(V)$, where S_i are certain matching cycles in E' , is remote with respect to E . The third ingredient - in §4.5 - is Seidel's exact triangle for which we provide a new proof reflecting our cobordism perspective. These ingredients are put together in §4.6. In short, the cobordism $V' = (\tau_{S_m} \circ \dots \circ \tau_{S_1})(V)$ is remote with respect to E and thus, by the first step, it admits a certain decomposition involving the ends of V , but as it is obtained by an iterated Dehn twist from V , it can be related to V by another decomposition, involving the matching cycles S_i , by using the relevant Seidel exact triangles. The two decompositions combine as in the statement of Theorem A.

The Corollaries of Theorem A described above are proven in §5.

The paper ends with §6 that consists of examples and related discussion. The main part of the section - §6.5 - is focused on a class of Lagrangian cobordisms in real Lefschetz fibrations.

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2. LEFSCHETZ FIBRATIONS

2.1. Basic definitions. Lefschetz fibrations will play a central role in this paper. From the symplectic viewpoint there are several versions of this notion in the literature. Our setup is similar to [Sei3, Sei2] but with some modifications.

We begin with Lefschetz fibrations having a compact fiber.

Definition 2.1.1. A Lefschetz fibration with compact fiber consists of the following data:

- i. A symplectic manifold (E, Ω_E) without boundary, endowed with a compatible almost complex structure J_E .
- ii. A Riemann surface (S, j) (which is generally not assumed to be compact; typically we will have $S = \mathbb{C}$).
- iii. A proper (J_E, j) -holomorphic map $\pi : E \rightarrow S$. (In particular all fibers of π are closed manifolds.)
- iv. We assume that π has a finite number of critical points. Moreover, we assume that every critical value of π corresponds to precisely one critical point of π . We denote the set critical points of π by $\text{Crit}(\pi)$ and by $\text{Critv}(\pi) \subset S$ the set of critical values of π . *Below we will use the words “critical points of π ” and “singularities of E ” interchangeably.*
- v. All the critical point of π are ordinary double points in the following sense. For every $p \in \text{Crit}(\pi)$ there exist a local J_E -holomorphic chart around p and a j -holomorphic chart around $\pi(p)$ with respect to which π is a holomorphic Morse function.

For $z \in S$ we denote by $E_z = \pi^{-1}(z)$ the fiber over z . We will sometimes fix a base-point $z_0 \in S \setminus \text{Critv}(\pi)$ and refer to the symplectic manifold $(M := \pi^{-1}(z_0), \omega_M := \Omega_E|_M)$ as “the” fiber of the Lefschetz fibration. We will also use the following notation: for a subset $\mathcal{S} \subset S$ we denote $V|_{\mathcal{S}} = \pi^{-1}(\mathcal{S}) \cap V$.

Our constructions work for the most part also when the fiber is not compact. To this end we will need some adjustments to the preceding definition as follows. Let (M, ω_M) be a

(non-compact) symplectic manifold which is convex at infinity. We define a *Lefschetz fibration* $\pi : E \longrightarrow S$ with fiber (M, ω_M) to be as in Definition 2.1.1 with the following modifications. Firstly, properness in condition iii is removed (thus allowing, in particular, for the fibers to be non-compact). Secondly, the map $\pi : E \setminus \pi^{-1}(\text{Critv}(\pi)) \longrightarrow S \setminus \text{Critv}(\pi)$ is now explicitly assumed to be a smooth locally trivial fibration. Finally, E is assumed to satisfy the following additional condition.

Assumption T_∞ (Triviality at infinity). *Let $\pi : E \longrightarrow S$ be as above. We say that E is trivial at infinity if there exists a subset $E^0 \subset E$ with the following properties:*

- (1) *For every compact subset $K \subset S$, $E^0 \cap \pi^{-1}(K)$ is also compact. (In other words, $\pi|_{E^0} \longrightarrow S$ is a proper map.)*
- (2) *Set $E^\infty = E \setminus E^0$ and $E_{z_0}^\infty = E^\infty \cap \pi^{-1}(z_0)$, where $z_0 \in S \setminus \text{Critv}(\pi)$ is a fixed base-point. Then there exists a trivialization $\phi : S \times E_{z_0}^\infty \longrightarrow E^\infty$ of $\pi|_{E^\infty} : E^\infty \longrightarrow S$ such that*

$$\phi^* \Omega_E = \omega_S \oplus \omega_M|_{E_{z_0}^\infty}, \text{ and } \phi^* J_E = j \oplus J_0$$

where ω_S is a positive (with respect to j) symplectic form on S and J_0 is a fixed almost complex structure on $M = \pi^{-1}(z_0)$, compatible with ω_M .

This extended definition in fact generalizes the preceding one: if M is compact we take $E^0 = E$ and $E^\infty = \emptyset$. From now on, unless otherwise stated, by a Lefschetz fibration we mean one with compact fiber that satisfies Definition 2.1.1 or, more generally, with a non-compact fiber that is convex at infinity and satisfies the conditions above, including T_∞ .

Before we go on, we recall again that *in this paper all Lefschetz fibrations are assumed to have positive dimensional fibers.*

Remark 2.1.2. a. The assumption that the fiber of E is either closed or symplectically convex was made in order to assure that the fiber is amenable to techniques of symplectic topology such as pseudo-holomorphic curves and Floer theory. (Specifically, these conditions assure that holomorphic curves and Floer trajectories cannot “escape to infinity”, hence standard compactness results hold for them.) Nevertheless in one instance later on in the paper we will drop this assumption and assume instead that M is itself the total space of another Lefschetz fibration.

b. Assumption T_∞ is a variant of boundary horizontality that appears in [Sei2] and [Sei3].

2.1.1. *Connections, parallel transport and trails of Lagrangians.* To a Lefschetz fibration as above we can associate a connection $\Gamma = \Gamma(\Omega_E)$ on $E \setminus \text{Crit}(\pi)$ as follows. The connection Γ is defined by setting its horizontal distribution $\mathcal{H} \subset T(E)$ to be the Ω_E -orthogonal complement of the tangent spaces to the fibers. More specifically, for every $x \in E \setminus \text{Crit}(\pi)$ we set

$$\mathcal{H}_x = \{u \in T_x(E) \mid \Omega_E(\xi, u) = 0 \ \forall \ \xi \in T_x^v(E)\},$$

where $T_x^v(E)$ stands for the vertical tangent space at x .

The connection Γ induces parallel transport maps. Let $\lambda : [a, b] \rightarrow \mathbb{C} \setminus \text{Critv}(\pi)$ be a smooth path. We denote by $\Pi_\lambda : E_{\lambda(a)} \rightarrow E_{\lambda(b)}$ the parallel transport along λ with respect to the connection Γ . Notice that even when the fiber of E is not compact, parallel transport is still well defined. Indeed, thanks to assumption T_∞ , the connection Γ is trivial at infinity with respect to the trivialization ϕ . In particular, relative to the trivialization ϕ , parallel transport becomes the identity at infinity in the sense that $\phi^{-1} \circ \Pi_\lambda \circ \phi(\lambda(a), x) = (\lambda(b), x)$ for every $x \in E_{z_0}^\infty$.

It is well known that Π_λ is a symplectomorphism, where we endow the fibers of π with the symplectic structure induced by Ω_E (See e.g. [MS2, Chapter 8], [MS1, Chapter 6].) If λ is a loop starting and ending at $z \in \mathbb{C} \setminus \text{Critv}(\pi)$ then the symplectomorphism $\Pi_\lambda : E_z \rightarrow E_z$ is also called the holonomy of Γ along λ . If the loop λ is contractible (within $\mathbb{C} \setminus \text{Critv}(\pi)$) then the holonomy Π_λ is in fact a Hamiltonian diffeomorphism of E_z (see [MS1, Section 6.4]).

Let $\lambda : [a, b] \rightarrow \mathbb{C} \setminus \text{Critv}(\pi)$ be a smooth embedding and $L \subset E_{\lambda(a)}$ a Lagrangian submanifold. Consider the images of L under the parallel transport along λ , namely $L_t := \Pi_{\lambda|_{[a, t]}}(L) \subset E_{\lambda(t)}$, $t \in [a, b]$ and set

$$\lambda L := \cup_{t \in [a, b]} L_t.$$

Then λL is a Lagrangian submanifold of (E, Ω_E) . We call λL the *trail* of L along λ .

We refer the reader to [Sei3] for the foundations of the symplectic theory of Lefschetz fibrations and to [MS1, Chapter 6] and [MS2, Chapter 8] for symplectic fibrations.

2.2. Lagrangians with cylindrical ends. Let $\pi : E \rightarrow \mathbb{C}$ be a Lefschetz fibration and $\mathcal{U} \subset \mathbb{C}$ an open subset containing $\text{Critv}(\pi)$. The following terminology is useful. A horizontal ray $\ell \subset \mathbb{C}$ is a half-line of the type $(-\infty, -a_\ell] \times \{b_\ell\}$ or $[a_\ell, \infty) \times \{b_\ell\}$ with $a_\ell > 0$, $b_\ell \in \mathbb{R}$. The imaginary coordinate b_ℓ is also referred to as the “height” of ℓ .

Definition 2.2.1. A Lagrangian submanifold (without boundary) $V \subset (E, \Omega_E)$ is said to have *cylindrical ends outside of \mathcal{U}* if the following conditions are satisfied:

- i. For every $R > 0$, the subset $V \cap \pi^{-1}([-R, R] \times \mathbb{R})$ is compact.
- ii. $\pi(V) \cap \mathcal{U}$ is bounded.
- iii. $\pi(V) \setminus \mathcal{U}$ consists of a finite union of horizontal rays, $\ell_i \subset \mathbb{C}$, $i = 1, \dots, r$. Moreover, for every i we have $V|_{\ell_i} = \ell_i L_i$ for some Lagrangian $L_i \subset E_{\sigma_i}$, where $\sigma_i \in \mathbb{C}$ stands for the starting point of the ray ℓ_i , and $\ell_i L_i$ is the trail of L_i along ℓ_i as defined above. (Note that we do allow $r = 0$, i.e. that V has no ends at all.)

In case all the heights of the rays ℓ_i are positive integers $b_{\ell_i} \in \mathbb{N}^*$ the Lagrangian V is called a *cobordism* in E .

In short, over each of the rays appearing in $\pi(V) \setminus \mathcal{U}$ the Lagrangian submanifold V is the trail under parallel transport of L_i along ℓ_i - see Figure 2.

The role of the condition ii above is to exclude the possibility that $\pi^{-1}(\mathcal{U})$ entirely covers some of the ends of V . For most of the time we will work with subsets \mathcal{U} that are U -shaped (see Figure 6 on page 27), and then condition ii is automatically satisfied (in view of condition i). However, occasionally we will have to consider \mathcal{U} 's that are not compact in the horizontal direction (see e.g. §4.4 and Figure 19), and then condition ii is necessary.

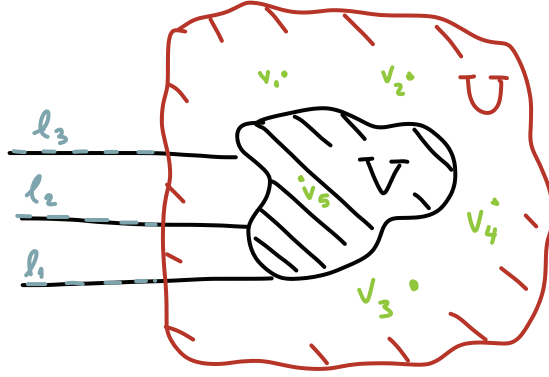


FIGURE 2. A Lagrangian V with cylindrical ends outside \mathcal{U} in a Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$ with critical values v_i .

The above notion of cobordism extends the definition of Lagrangian cobordism as given for the trivial fibration E in [BC2]. Note however that this terminology is slightly imprecise because we have not specified a (topological) trivialization of the fibration $E \rightarrow \mathbb{C}$ at infinity (and in general there is no canonical trivialization). Moreover, even when one fixes such a trivialization the parallel transport along a ray ℓ_i might not be trivial (even not at infinity), hence the actual ends of V at infinity are not well defined. In view of that, we will often work with a restricted type of Lefschetz fibrations, called tame, where this imprecision is not present and that have a number of additional technical advantages. We will see later on that this does not restrict the generality of our theory.

Definition 2.2.2. Let $\pi : E \rightarrow \mathbb{C}$. Let $U \subset \mathbb{C}$ be a closed subset, let $z_0 \in \mathbb{C} \setminus U$ be a base point and (M, ω_M) be the fiber over z_0 . We say that this Lefschetz fibration is tame outside of U if there exists a trivialization

$$\psi_{E, \mathbb{C} \setminus U} : (\mathbb{C} \setminus U) \times M \rightarrow E|_{\mathbb{C} \setminus U}$$

such that $\psi_{E, \mathbb{C} \setminus U}^*(\Omega_E) = c\omega_{\mathbb{C}} \oplus \omega_M$, where $\omega_{\mathbb{C}}$ is the standard symplectic structure on $\mathbb{C} \cong \mathbb{R}^2$ and $c > 0$ is a constant. The manifold (M, ω_M) is called the generic fiber of π .

It follows from the definition that all the critical values of π must be contained inside U . Sometimes it will be more natural to fix the complement of U , say $\mathcal{W} = \mathbb{C} \setminus U$, and say that the fibration is tame over \mathcal{W} . Given a tame Lefschetz fibration, the set $U = U_E$, the point z_0 and the symplectic trivialization $\psi_{E, \mathbb{C} \setminus B}$, are all viewed as part of the fixed data associated to the fibration.

Moreover, we will assume that the set $U = U_E$ is so that there exists $a_U > 0$ sufficiently large with the property that U is disjoint from both quadrants:

$$(4) \quad Q_U^- = (-\infty, -a_U] \times [0, +\infty), \quad Q_U^+ = [a_U, \infty) \times [0, +\infty)$$

The cobordism relation, as defined in [BC2], admits an obvious extension in a tame Lefschetz fibration.

Definition 2.2.3. Fix a Lefschetz fibration that is tame outside $U \subset \mathbb{C}$ with fiber (M, ω) over $z_0 \in \mathbb{C} \setminus U$. Let $(L_i)_{1 \leq i \leq k_-}$ and $(L'_j)_{1 \leq j \leq k_+}$ be two families of closed Lagrangian submanifolds of M . We say that these two families are Lagrangian cobordant in E , if there exists a Lagrangian submanifold $V \subset E$ with the following properties:

- i. There is a compact set $K \subset E$ so that $V \cap U \subset V \cap K$ and $V \setminus K \subset \pi^{-1}(Q_U^+ \cup Q_U^-)$.
- ii. $V \cap \pi^{-1}(Q_U^+) = \coprod_j ([a_U, +\infty) \times \{j\}) \times L'_j$
- iii. $V \cap \pi^{-1}(Q_U^-) = \coprod_i ((-\infty, -a_U] \times \{i\}) \times L_i$

The formulas at ii and iii are written with respect to the trivialization of the fibration over the complement of U .

The manifold V is obviously a Lagrangian cobordism in the sense of Definition 2.2.1 and - because of tameness - its ends at ∞ are well defined so that we can say that V is a cobordism from the Lagrangian family (L'_j) to the family (L_i) . We write $V : (L'_j) \rightsquigarrow (L_i)$ or $(V; (L_i), (L'_j))$.

2.3. From general Lefschetz fibrations to tame ones. We will now see that it is always possible to pass from a general Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$, as in §2.1, to a tame one.

Proposition 2.3.1. *Let $\pi : E \rightarrow \mathbb{C}$ be a Lefschetz fibration and let $\mathcal{N} \subset \mathbb{C}$ be an open subset that contains all the critical values of π and has the shape depicted in Figure 3. Let $\mathcal{W} \subset \mathbb{C}$ be another open subset of the shape depicted in Figure 3 with $\overline{\mathcal{W}} \cap \overline{\mathcal{N}} = \emptyset$ and $\text{dist}(\overline{\mathcal{W}}, \overline{\mathcal{N}}) > 0$. Then there exists a symplectic structure $\Omega' = \Omega'_{E, \mathcal{N}, \mathcal{W}}$ on E and a trivialization $\varphi : \mathcal{W} \times M \rightarrow E|_{\mathcal{W}}$ with the following properties:*

- (1) On $\mathcal{W} \times M$ we have $\varphi^* \Omega' = c\omega_{\mathbb{C}} \oplus \omega_M$ for some $c > 0$.
- (2) Ω' coincides with Ω_E on all the fibers of π .
- (3) $\Omega' = \Omega_E$ on $\pi^{-1}(\mathcal{N})$.

- (4) *There exists an Ω' -compatible almost complex structure J'_E on E which coincides with J_E on $\pi^{-1}(\mathcal{N})$ and such that the projection $\pi : E \rightarrow \mathbb{C}$ is (J'_E, i) -holomorphic.*

In particular, when endowed with the symplectic structure Ω' , the Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$ is tame over \mathcal{W} .

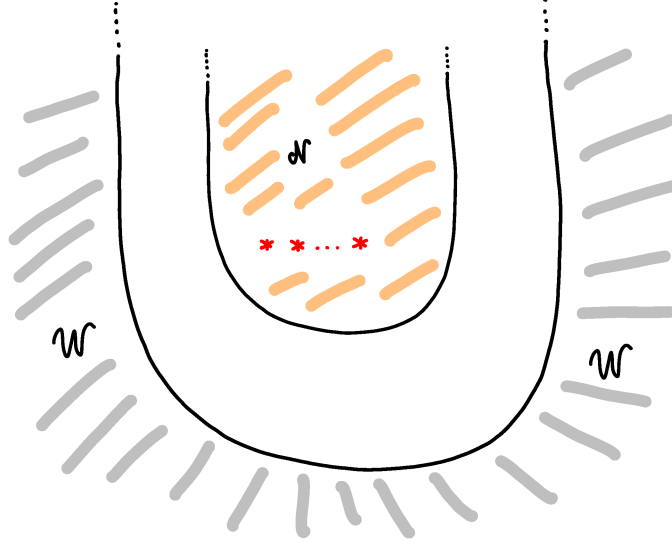


FIGURE 3. A Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$; the domains \mathcal{N} and \mathcal{W} and, in red, the critical values of π .

Remark 2.3.2. It is easy to pass from a cobordism in a general Lefschetz fibration to a cobordism in a tame fibration.

Indeed, let $\pi : E \rightarrow \mathbb{C}$ be a Lefschetz fibration and $V \subset E$ a Lagrangian submanifold with cylindrical ends. Let $\mathcal{N} \subset \mathbb{C}$ be a subset as in Proposition 2.3.1 and assume that V has cylindrical ends outside of \mathcal{N}' , where $\mathcal{N}' \subset \mathcal{N}$ is a slightly smaller subset than \mathcal{N} which contains $\text{Critv}(\pi)$ and is of the same shape as \mathcal{N} . Denote the horizontal rays corresponding to the ends of V by $\ell_i \subset \mathbb{C}$, $i = 1, \dots, r$ and by $L_i \subset E_{\sigma_i}$ the corresponding Lagrangians over the starting points of these rays. Let $\mathcal{W} \subset \mathbb{C}$ be a subset as in Proposition 2.3.1 and consider the new symplectic structure Ω' on E provided by that proposition. By performing parallel transport of the L_i 's along the horizontal rays ℓ_i , but this time with respect to the connection corresponding to (E, Ω') we obtain a new Lagrangian submanifold $V' \subset (E, \Omega')$ with the following properties:

- i. V' coincides with V over \mathcal{N} .
- ii. V' has cylindrical ends outside of \mathcal{N} .

iii. Over \mathcal{W} , V' looks like

$$V'|_{\mathcal{W}} = \cup_{i=1}^r \ell'_i \times L'_i,$$

where $\ell'_i = \ell_i \cap \mathcal{W}$ and L'_i is the image of the parallel transport of L_i (with respect to the connection $\Gamma(\Omega')$) along the portion of ℓ_i that connects \mathcal{N}' with \mathcal{W} .

2.3.1. Preparation for the proof of Proposition 2.3.1. Let (M, ω) be a symplectic manifold, $Q \subset \mathbb{C}$ an open subset and $f : Q \times M \rightarrow \mathbb{R}$ a smooth function. We denote by $z = y_1 + iy_2$ the standard complex coordinate in \mathbb{C} . Let $\alpha = \{\alpha_z\}_{z \in Q}, \beta = \{\beta_z\}_{z \in Q}$ be two families of 1-forms on M , parametrized by $z \in Q$ (alternatively we can view α, β as differential forms on $Q \times M$ with $\alpha(\frac{\partial}{\partial y_j}) = \beta(\frac{\partial}{\partial y_j}) = 0$). For $z \in Q, p \in M$ we write $\alpha_{z,p}$ for the restriction of α_z to $T_p(M)$ and similarly for β . We denote by d^v the exterior derivative of differential forms on $Q \times M$ in the M -direction (i.e. $(d^v \alpha)_z = d^M(\alpha_z)$, where d^M is the exterior derivative in M .) Below we will abbreviate the partial derivatives $\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}$ by $\partial_{y_1}, \partial_{y_2}$.

Consider now the following 2-form on $Q \times M$

$$\Omega^{f,\alpha,\beta} := \omega + f dy_1 \wedge dy_2 + \alpha \wedge dy_1 + \beta \wedge dy_2.$$

A simple calculation shows that:

Lemma 2.3.3. $\Omega^{f,\alpha,\beta}$ is closed iff $d^v \alpha = d^v \beta = 0$ and $d^v f = \partial_{y_2} \alpha - \partial_{y_1} \beta$.

Define now two families of vector fields u_0, v_0 on M (parametrized by the points of Q) as follows. For every $z \in Q, p \in M$, define $u_0(z, p), v_0(z, p) \in T_p(M)$ by requiring that for every $\xi \in T_p(M)$ we have:

$$(5) \quad \omega_p(\xi, u_0(z, p)) + \alpha_{z,p}(\xi) = 0, \quad \omega_p(\xi, v_0(z, p)) + \beta_{z,p}(\xi) = 0.$$

Denote by $\mathcal{H} \subset T(Q \times M)$ the following 2-dimensional distribution:

$$(6) \quad \mathcal{H}_{z,p} := \mathbb{R} \left(\frac{\partial}{\partial y_1} + u_0(z, p) \right) + \mathbb{R} \left(\frac{\partial}{\partial y_2} + v_0(z, p) \right).$$

Note that \mathcal{H} depends on ω, α, β but not on f .

The following two lemmas can be proved by direct calculation.

Lemma 2.3.4. For every $(z, p) \in Q \times M, \xi \in T_p(M)$ and $w \in \mathcal{H}_{z,p}$ we have $\Omega^{f,\alpha,\beta}(\xi, w) = 0$. In particular, if $\Omega^{f,\alpha,\beta}$ is non-degenerate then \mathcal{H} is the horizontal distribution of the connection induced by $\Omega^{f,\alpha,\beta}$.

Lemma 2.3.5. Assume that $f(z, p) \neq \omega_p(u_0(z, p), v_0(z, p))$ for some $(z, p) \in Q \times M$. Then $\Omega^{f,\alpha,\beta}$ is non-degenerate at (z, p) . Moreover, there exists an $\Omega_{z,p}^{f,\alpha,\beta}$ -compatible complex structure $J_{z,p}$ on $T_{z,p}(Q \times M)$ such that the projection $Q \times M \rightarrow Q$ is $(J_{z,p}, i)$ -holomorphic at (z, p) if and only if $f(z, p) > \omega_p(u_0(z, p), v_0(z, p))$.

2.3.2. Proof of Proposition 2.3.1.

To fix ideas, we first provide the proof in the case of compact fibre.

Step 1. Using parallel transport with respect to the connection Γ_{Ω_E} along a system of curves in $\mathbb{C} \setminus \overline{\mathcal{N}}$ emanating from a fixed point $z_0 \in \mathcal{W}$, and using the fact that $\mathbb{C} \setminus \overline{\mathcal{N}}$ is contractible we obtain a trivialization

$$\varphi : (\mathbb{C} \setminus \overline{\mathcal{N}}) \times M \longrightarrow E|_{\mathbb{C} \setminus \overline{\mathcal{N}}}$$

with $M = \pi^{-1}(z_0)$ and with the property that the form $\Omega_1 := \varphi^* \Omega_E$ admits the following form

$$(7) \quad \Omega_1 = f dy_1 \wedge dy_2 + \alpha \wedge dy_1 + \beta \wedge dy_2 + \omega,$$

where $\omega = \Omega|_M$ and $f : (\mathbb{C} \setminus \overline{\mathcal{N}}) \times M \longrightarrow \mathbb{R}$ is a smooth function, and α, β are vertical 1-forms on $(\mathbb{C} \setminus \overline{\mathcal{N}}) \times M$ with the property that for every $z \in \mathbb{C} \setminus \overline{\mathcal{N}}$ the 1-forms $\alpha_z = \alpha|_{z \times M}$, $\beta_z = \beta|_{z \times M}$ are exact (see § 8.2 of [MS2] and § 6.4 of [MS1] for a proof of that). Fix two functions $F, G : (\mathbb{C} \setminus \overline{\mathcal{N}}) \times M \longrightarrow \mathbb{R}$ such that $\alpha = d^v F$, $\beta = d^v G$.

By Lemma 2.3.3 we have:

$$(8) \quad d^v f = \partial_{y_2} \alpha - \partial_{y_1} \beta.$$

Apart from \mathcal{W} and \mathcal{N} we will fix three additional open subsets $\mathcal{W}_\epsilon, \mathcal{N}_\epsilon, \mathcal{N}_{2\epsilon}$ with

$$\overline{\mathcal{W}} \subset \mathcal{W}_\epsilon, \quad \overline{\mathcal{N}} \subset \mathcal{N}_\epsilon, \quad \overline{\mathcal{N}}_\epsilon \subset \mathcal{N}_{2\epsilon},$$

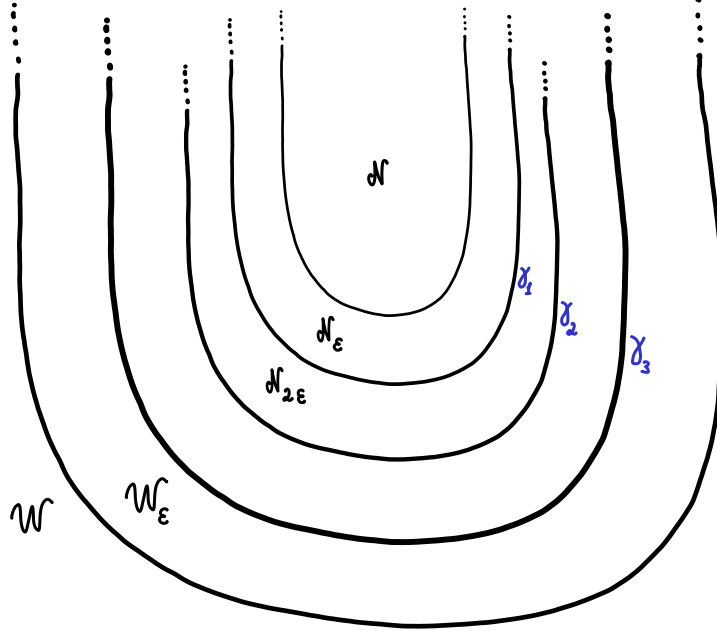
and with shapes as described in Figure 4. To be more precise, consider the curves $\gamma_1, \gamma_2, \gamma_3 \subset \mathbb{C}$ depicted in Figure 4. The domain \mathcal{N}_ϵ is defined to be the connected component of $\mathbb{C} \setminus \gamma_1$ in which all the points have bounded real coordinate. The domain $\mathcal{N}_{2\epsilon}$ is defined similarly but with the curve γ_1 replaced by γ_2 . The domain \mathcal{W}_ϵ is defined as the connected component of $\mathbb{C} \setminus \gamma_3$ in which the real coordinate of the points is unbounded. We also require that $\text{dist}(\overline{\mathcal{W}}_\epsilon, \overline{\mathcal{N}}_{2\epsilon}) > 0$.

Step 2. We will modify now the form Ω_1 in the following way. Fix a smooth function $\sigma : \mathbb{C} \longrightarrow [0, 1]$ such that:

$$(9) \quad \sigma(z) = \begin{cases} 1 & z \in \mathcal{N}_{2\epsilon}, \\ 0 & z \in \mathcal{W}_\epsilon. \end{cases}$$

Define $g : \mathbb{C} \times M \longrightarrow \mathbb{R}$ by

$$(10) \quad g(z, p) = \partial_{y_2}(\sigma)F(z, p) - \partial_{y_1}(\sigma)G(z, p).$$

FIGURE 4. The domains \mathcal{N}_ϵ , $\mathcal{N}_{2\epsilon}$, and \mathcal{W}_ϵ .

Then we have:

$$(11) \quad g(z, p) = 0 \quad \forall z \in \mathcal{N}_{2\epsilon} \cup \mathcal{W}_\epsilon.$$

Next, choose a function $A : \mathbb{C} \longrightarrow \mathbb{R}$ with the following properties:

- (A.1) $A(z) \geq 0$ for every $z \in \mathbb{C}$.
- (A.2) $A(z) = 0$ for every $z \in \mathcal{N}_\epsilon$.
- (A.3) $A(z) \geq |g(z, p)|$ for every $z \in \mathbb{C}$, $p \in M$.
- (A.4) Let u_0, v_0 be the vector fields associated to the form $\Omega_1 = \Omega^{f, \alpha, \beta}$ from (7) using the recipe from (5). We require that

$$A(z) > \sigma(z) |f(z, p) - \sigma(z) \omega_p(u_0(z, p), v_0(z, p))| + |g(z, p)|$$

for every $z \in \mathbb{C} \setminus \mathcal{N}_{2\epsilon}$, $p \in M$.

- (A.5) $A(z) = C$ for every $z \in \mathcal{W}$, for some constant $C > 0$.

The role of the function A is to flatten the form Ω_1 on \mathcal{W} , so it is split there, while ensuring non-degeneracy. Such a function A can be constructed as follows. We start by defining a function $A' : \mathbb{C} \longrightarrow \mathbb{R}$ which is positive and satisfies condition (A.4) (with $A'(z)$ on the left-hand side of the inequality). Such a function obviously exists because M is compact. We then cut A' off to make it 0 on \mathcal{N}_ϵ and constant on \mathcal{W} , where the cutting off takes place within $\mathcal{N}_{2\epsilon} - \mathcal{N}_\epsilon$ and within $\mathcal{W}_\epsilon - \mathcal{W}$, where the function g is 0 anyway. It is easy to see that the cutting off can be done in such that the inequality in (A.4) continues to hold and similarly

for (A.3). The function resulting from A' after this procedure can be taken to be the desired function A . See Figure 5.

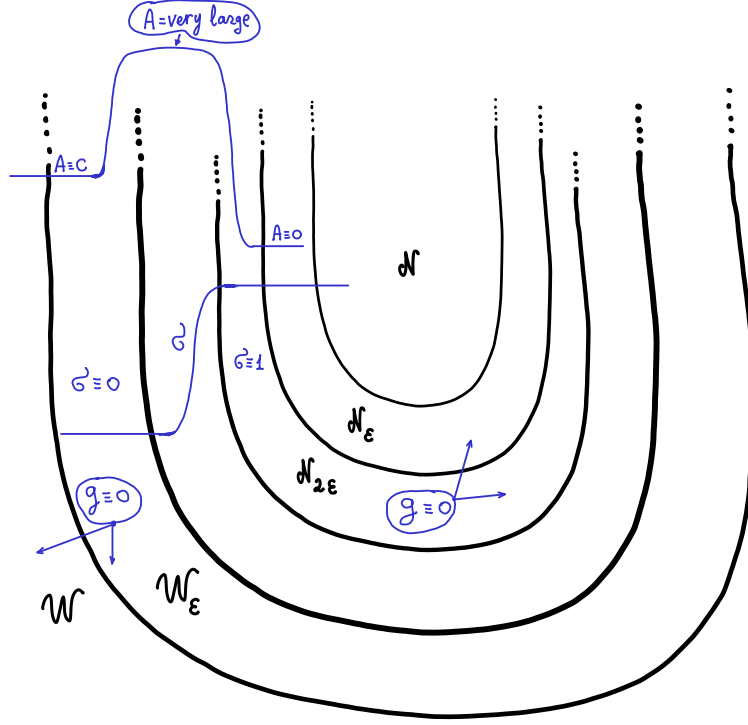


FIGURE 5. The functions σ , A and g .

Finally, define:

$$\begin{aligned}
 f'(z, p) &:= \sigma(z)f(z, p) + g(z, p) + A(z), \\
 \alpha'_{z,p} &:= \sigma(z)\alpha_{z,p} = d^v(\sigma(z)F)_{z,p}, \\
 \beta'_{z,p} &:= \sigma(z)\beta_{z,p} = d^v(\sigma(z)G)_{z,p}.
 \end{aligned}
 \tag{12}$$

Consider now the form

$$\Omega_2 := \Omega^{f', \alpha', \beta'} = f' dy_1 \wedge dy_2 + \alpha' \wedge dy_1 + \beta' \wedge dy_2 + \omega.
 \tag{13}$$

Note that Ω_2 coincides with Ω_1 over a small neighborhood of $\overline{\mathcal{N}}$ and therefore Ω_2 gives rise via the trivialization φ to a well defined 2-form Ω' over the whole of E . Moreover Ω' coincides with Ω on $\pi^{-1}(\mathcal{N})$.

We claim that Ω' is a symplectic form on E and that it satisfies all the properties claimed by Proposition 2.3.1.

We first show that Ω_2 is closed using Lemma 2.3.3. Indeed

$$\begin{aligned} d^v f' &= \sigma d^v f + d^v g = \sigma \partial_{y_2}(\alpha) - \sigma \partial_{y_1}(\beta) + d^v g \\ &= \partial_{y_2}(\sigma \alpha) - \partial_{y_1}(\sigma \beta) + (d^v g - \partial_{y_2}(\sigma) \alpha + \partial_{y_1}(\sigma) \beta) \\ &= \partial_{y_2}(\alpha') - \partial_{y_1}(\beta'). \end{aligned}$$

Here the last term (between the brackets) on the second line vanishes by (10).

We now prove that Ω_2 is non-degenerate and moreover admits a compatible almost complex structure J' for which the projection $\mathbb{C} \times M \rightarrow \mathbb{C}$ is (J', i) -holomorphic. Note that with the notation from (5) and (6) the effect of replacing α and β by $\alpha' = \sigma \alpha$ and $\beta' = \sigma \beta$ results in changing the vector fields u_0, v_0 to $u'_0 = \sigma u_0$, $v'_0 = \sigma v_0$. Thus by Lemma 2.3.5 we only need to check that:

$$(14) \quad f'(z, p) > \omega_p(u'_0(z, p), v'_0(z, p)) \quad \forall p \in M, z \in \mathbb{C} \setminus \overline{\mathcal{N}}.$$

We have:

$$\begin{aligned} (15) \quad f'(z, p) - \omega_p(u'_0(z, p), v'_0(z, p)) &= \sigma(z) f(z, p) + g(z, p) + A(z) - \sigma^2(z) \omega_p(u_0, v_0) \\ &= \sigma(z) (f(z, p) - \sigma(z) \omega_p(u_0, v_0)) + (g(z, p) + A(z)). \end{aligned}$$

We denote by $T_1 = \sigma(z) (f(z, p) - \sigma(z) \omega_p(u_0, v_0))$ the first term on the last line of (15) and by $T_2 = g(z, p) + A(z)$ the second one.

We first verify (14) over $\pi^{-1}(\mathcal{W}_\epsilon)$. Indeed, when $z \in \mathcal{W}_\epsilon$ we have $\sigma(z) = 0$ hence $T_1 = 0$. By the construction of the function A we have $T_2 > 0$, hence $T_1 + T_2 > 0$.

Next we check (14) over $\pi^{-1}(\mathcal{N}_{2\epsilon} \setminus \overline{\mathcal{N}})$. Let $z \in \mathcal{N}_{2\epsilon} \setminus \overline{\mathcal{N}}$ and $p \in M$. Note that $\sigma(z) = 1$ hence $T_1 = f(z, p) - \omega_p(u_0(z, p), v_0(z, p)) > 0$ by Lemma 2.3.5. Since $T_2 \geq 0$ we have $T_1 + T_2 > 0$.

Finally, the inequality (14) for $z \in \mathbb{C} \setminus (\mathcal{N}_{2\epsilon} \cup \mathcal{W}_\epsilon)$ follows easily from requirement (A.4) in the construction of the function A .

To finish the proof, we turn to the case of a non-compact fibre. Thus we assume the conditions in §2.1 and, in particular, assumption T_∞ . The proof above applies in this case too, and we will preserve all the notation above, but there are a number of adjustments that we describe below. Recall the set E^∞ that appears in the assumption T_∞ and put $M^\infty = M \cap E^\infty$. Recall also that, as before, $M = \pi^{-1}(z_0)$. Let

$$\phi : \mathbb{C} \times M^\infty \rightarrow E^\infty$$

be the trivialization provided by T_∞ . Consider also the restriction of this trivialization to $\mathbb{C} \setminus \overline{\mathcal{N}}$:

$$(16) \quad \phi : (\mathbb{C} \setminus \overline{\mathcal{N}}) \times M^\infty \rightarrow E^\infty|_{\mathbb{C} \setminus \overline{\mathcal{N}}}$$

and put $\phi_0 : M^\infty \rightarrow M^\infty$, $\phi_0(p) = \phi(z_0, p)$.

Consider also the map φ constructed at the **Step 1** above and its restriction:

$$\varphi : (\mathbb{C} \setminus \overline{\mathcal{N}}) \times M^\infty \rightarrow E^\infty|_{\mathbb{C} \setminus \overline{\mathcal{N}}}$$

which is well defined due to Assumption T_∞ .

For brevity, write $\Omega = \Omega_E$. Given that the connection associated to $\phi^*\Omega$ is trivial on $(\mathbb{C} \setminus \overline{\mathcal{N}}) \times M^\infty$, we deduce that $\varphi(z, p) = \phi(z, \phi_0^{-1}(p))$ for all $z \in \mathbb{C} \setminus \overline{\mathcal{N}}$, $p \in M^\infty$. Therefore $\phi^*\Omega|_{(\mathbb{C} \setminus \overline{\mathcal{N}}) \times M^\infty} = \omega_{\mathbb{C}} \oplus \omega$.

Recall that over $(\mathbb{C} \setminus \overline{\mathcal{N}}) \times M$ the form $\Omega_1 = \varphi^*\Omega$ can be written as

$$\Omega_1 = \omega + \alpha \wedge dy_1 + \beta \wedge dy_2 + f dy_1 \wedge dy_2 .$$

This means that α, β vanish over $(\mathbb{C} \setminus \overline{\mathcal{N}}) \times M^\infty$ and f is constant there. Therefore, we can choose the functions F, G so that they both vanish on $(\mathbb{C} \setminus \overline{\mathcal{N}}) \times M^\infty$. Starting from this point the remainder of the proof continues as in the compact fibre case by using the fact that $g(z, p)$, as well as $\alpha', \beta', u_0(z, p), v_0(z, p)$ all vanish over $(\mathbb{C} \setminus \overline{\mathcal{N}}) \times M^\infty$.

Recall now the forms Ω_2 and Ω' (defined by formula (13) and the paragraph following it). Summing up the preceding discussion, the form Ω_2 hence also Ω' satisfies $\phi^*\Omega' = B(z)\omega_{\mathbb{C}} \oplus \omega$ over $\mathbb{C} \times M^\infty$, where $B(z)$ is positive and bounded. By adding to Ω' another term of the form $D(z)\pi^*\omega_{\mathbb{C}}$ we obtain a form that satisfies all the properties claimed in Proposition 2.3.1 as well as the assumption T_∞ . (The role of adding the last term is to ensure that property (1) in Proposition 2.3.1 is satisfied.) \square

3. FUKAYA CATEGORIES

The purpose of this section is to introduce the various Fukaya categories that play a role in the paper. We start with a brief sketch of the construction of the Fukaya category $\mathcal{Fuk}^*(M)$ of uniformly monotone, closed Lagrangian submanifolds of a symplectic manifold (M, ω) which is assumed to be either closed or convex at infinity. The full construction in the exact case can be found in [Sei3, Sections 8-12] (the minor adjustments required in the monotone case are described, for instance, in [BC3]). In §3.3, we pursue with the construction of the Fukaya category $\mathcal{Fuk}^*(E)$ of uniformly monotone cobordisms in a tame Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$ of generic fiber (M, ω) . This follows closely §3 of [BC3] where this construction is implemented for the trivial fibration $E = \mathbb{C} \times M$. The passage from a trivial fibration to a tame one is quite straightforward but we provide enough details on this construction as required for further arguments later in the paper and also to ensure that the notions involved are accessible to a reader without prior detailed knowledge of the techniques in [BC3]. In §3.4 we use the construction in the tame setting together with the results in §2.3 to define a Fukaya category associated to a general Lefschetz fibration.

In the definition of the various algebraic objects used in the paper there are two coefficient rings of interest, \mathbb{Z}_2 and the universal Novikov ring \mathcal{A} over \mathbb{Z}_2 :

$$\mathcal{A} = \left\{ \sum_{k=0}^{\infty} a_k T^{\lambda_k} : a_k \in \mathbb{Z}_2, \lambda_k \in \mathbb{R}, \lim_{k \rightarrow \infty} \lambda_k \rightarrow \infty \right\}.$$

We work over \mathcal{A} at all times except if otherwise indicated.

3.1. The Fukaya category of M . The main structures in use in the paper are the Fukaya category, $\mathcal{Fuk}^*(-)$, and the derived Fukaya category, $D\mathcal{Fuk}^*(-)$. Here $*$ encodes a uniform monotonicity constraint imposed to the objects of $\mathcal{Fuk}^*(M)$. This constraint is necessary to define the A_∞ -operations.

The book [Sei3] is a comprehensive reference for the basic definitions of the A_∞ machinery as well as the construction of the Fukaya category and its derived version. Our notation - which is homological¹, in contrast to Seidel's which is cohomological - is the same as in [BC3], see in particular the Appendix to that paper. There is a single difference with respect to [BC3] which is that we use here the universal Novikov ring \mathcal{A} in the place of \mathbb{Z}_2 . As we shall see, this is not a matter of choice, rather a requirement for a certain part of our results to hold. We emphasize that in the construction of $D\mathcal{Fuk}^*(-)$ we do not complete with respect to idempotents. Moreover, as in [BC3] we work in an ungraded context.

Fix a symplectic manifold (M, ω) , compact or convex at infinity. Given a closed Lagrangian submanifold $L \subset M$ there are two morphisms

$$\mu : \pi_2(M, L) \rightarrow \mathbb{Z}, \quad \omega : \pi_2(M, L) \rightarrow \mathbb{R}$$

given, the first, by the Maslov index and, the second, by integration of ω . We say that L is monotone if $\omega(\alpha) = \rho\mu(\alpha)$ for some constant $\rho \geq 0$ and if the number

$$N_L = \min\{\mu(\alpha) : \alpha \in \pi_2(M, L), \omega(\alpha) > 0\}$$

is at least 2.

Note that we do allow $\rho = 0$ in the definition of monotonicity. This means that ω vanishes on $\pi_2(M, L)$ (such Lagrangians are sometime called weakly exact). In this case we set $N_L = \infty$.

For a connected monotone Lagrangian L and for a generic almost complex structure J compatible with ω , the number (mod 2) of J -holomorphic disks of Maslov number 2 that pass through a generic point of L is an invariant (in the sense that it does not depend either on

¹Since we work in an ungraded setting, the difference between homological and cohomological might seem invisible. However, our Floer homologies correspond to Morse homology rather than cohomology. In particular the unity in $HF(L, L)$ corresponds to the fundamental class of L etc. Apart from that, the ordering of the terms in the higher operations μ_k is opposite to Seidel's and our conventions for the Yoneda embedding differs from Seidel's. This is all described in detail in the Appendix to [BC3].

the point or on the choice of J). It is denoted by d_L (and is defined in detail, for instance, in [BC1]). Note that in case $\rho = 0$ we set $d_L = 0$ by definition.

In order to define the Fukaya category of M we first need to specify its underlying class of Lagrangian submanifolds. In what follows we will mainly consider two classes of Lagrangians $\mathcal{L}^{(0)}(M)$ and $\mathcal{L}^{(\rho,1)}(M)$, which are defined as follows:

- a. The class $\mathcal{L}^{(0)}(M)$: this class consists of all closed monotone Lagrangians $L \subset M$ with $d_L = 0$. This includes in particular all Lagrangians with $N_L \geq 3$ as well as the case $\rho = 0$.
- b. Class $\mathcal{L}^{(\rho,1)}(M)$: consists of all the closed monotone Lagrangians $L \subset M$ with $d_L = 1$ and with monotonicity constant ρ , where $\rho > 0$ is a prescribed positive real number.

Of course one could restrict also to some subclasses of the above. For example, when M is exact it makes sense to restrict to the subclass $\mathcal{L}^{(\text{ex})}(M) \subset \mathcal{L}^{(0)}(M)$ of exact Lagrangian submanifolds.

To simplify the notation will denote any of these two choices by $\mathcal{L}^*(M)$, where the symbol $*$ stands for either (0) in the first case, or for $(\rho, 1)$ in the second case. Lagrangians in the class $\mathcal{L}^*(M)$ will be called *uniformly monotone* of class $*$.

In what follows we will work also with uniformly monotone *negatively-ended* Lagrangian cobordisms in the total space of a Lefschetz fibration $E \rightarrow \mathbb{C}$. Similarly to the Lagrangians in M we will denote the various classes of uniformly monotone Lagrangian cobordisms in E by $\mathcal{L}^*(E)$, where the definition of these classes is the same as above except that the Lagrangians in E are not assumed to be compact.

Floer homology will be taken in this paper with coefficients in the Novikov ring \mathcal{A} and its definition will be shortly reviewed below. It was introduced by Floer in [Flo] and, in this monotone setting, by Oh [Oh1, Oh2].

- Remarks.*
- a. In contrast to [BC3] there is no injectivity condition on the inclusions $\pi_1(L) \rightarrow \pi_1(M)$ (this is because the coefficient ring is \mathcal{A} and not \mathbb{Z}_2).
 - b. In case there exists a spherical class $A \in \pi_2(M)$ with $\omega(A) > 0$, the monotonicity constant ρ is determined by the proportionality constant between $[\omega]$ and the first Chern class of the ambient symplectic manifold. Thus in this case there is only one class of the type $\mathcal{L}^{(\rho,1)}$.

The Fukaya A_∞ -category $\mathcal{Fuk}^*(M)$ has as objects the Lagrangians in $\mathcal{L}^*(M)$,

$$\mathcal{Ob}(\mathcal{Fuk}^*(M)) = \mathcal{L}^*(M) .$$

Let $L, L' \in \mathcal{L}^*(M)$ and assume for the moment that L and L' intersect transversely. In this case, the Floer complex, $(CF(L, L'; J), d)$, associated to L and L' is defined by choosing a

regular almost complex structure J compatible with ω and is a free \mathcal{A} -module with generators the intersection points of L and L' . In this paper $CF(L, L')$ is a complex without grading.

The differential d is defined in terms of J -holomorphic strips $u : \mathbb{R} \times [0, 1] \rightarrow M$ with $u(\mathbb{R} \times \{0\}) \subset L$, $u(\mathbb{R} \times \{1\}) \subset L'$ and $\lim_{s \rightarrow -\infty} u(s, t) = x \in L \cap L'$, $\lim_{s \rightarrow +\infty} u(s, t) = y \in L \cap L'$. We have:

$$d(x) = \sum_y \sum_{u \in \mathcal{M}_0(x, y)} T^{\omega(u)} y$$

where the sum is over all the intersection points $y \in L \cap L'$ and $\mathcal{M}_0(x, y)$ is the 0-dimensional subspace of the moduli space of J -strips u joining x to y . Uniform monotonicity is used to show that $d^2 = 0$.

The homology of this complex, $HF(L, L')$, is the Floer homology of L and L' . It is independent of J as well as of Hamiltonian perturbation of L and of L' .

The morphisms in $\mathcal{Fuk}^*(M)$ are $\text{Mor}_{\mathcal{Fuk}^*(M)}(L, L') = CF(L, L')$. The A_∞ structural maps are, by the definition of an A_∞ -category, multilinear maps

$$\mu_k : CF(L_1, L_2) \otimes CF(L_2, L_3) \otimes \dots \otimes CF(L_k, L_{k+1}) \rightarrow CF(L_1, L_{k+1})$$

that satisfy the relation $\mu \circ \mu = \sum \mu(-, -, \dots, \mu, \dots, -, -) = 0$. In our case, these maps are such that $\mu_1 = d$ = the Floer differential and, for $k > 1$, μ_k is defined by:

$$(17) \quad \mu_k(x_1, \dots, x_k) = \sum_y \sum_{u \in \mathcal{M}_0(x_1, \dots, x_k; y)} T^{\omega(u)} y.$$

Here, at least when the L_i 's and L are in general position, $x_i \in L_i \cap L_{i+1}$, $y \in L_1 \cap L_{k+1}$ and $\mathcal{M}_0(x_1, \dots, x_k; y)$ is the 0-dimensional moduli space of (perturbed) J -holomorphic polygons with $k + 1$ sides that have k “inputs” asymptotic - in order - to the intersection points x_i and one “exit” asymptotic to y . Monotonicity is used to show that the sums in (17) are well defined over \mathcal{A} . The relation $\mu \circ \mu = 0$ extends the relation $d^2 = 0$.

This is just a rough summary of the construction as, in particular, the operations μ_k have to be defined for all families L_1, \dots, L_{k+1} and not only when L_i, L_{i+1} , etc., are transverse. In reality one has to add perturbation terms to the Cauchy-Riemann equation that come from Hamiltonian functions associated to each vertex of the polygon and the asymptotic conditions x_i, y , are replaced by trajectories γ_i, γ of the flows of these Hamiltonian functions that start on L_i and end on L_{i+1} , respectively start on L_1 and end on L_k . Moreover, the regularity of these moduli spaces depends on a number of choices of auxiliary data, basically a coherent system of *strip-like ends* and coherent *perturbation data*. We refer to [Sei3] for the actual implementation of the construction which is considerably more involved. Additionally, these notions are made more precise in §3.3 where we discuss in more detail some of the ingredients used in the construction of a Fukaya category $\mathcal{Fuk}^*(E)$ with objects certain cobordisms in E .

Consider next the category of A_∞ -modules over the Fukaya category

$$\text{mod}(\mathcal{Fuk}^*(M)) := \text{fun}(\mathcal{Fuk}^*(M), Ch^{opp})$$

where Ch^{opp} is the opposite of the dg-category of chain complexes over \mathcal{A} . The category of A_∞ -modules is an A_∞ -category in itself (in fact a dg-category) and is triangulated in the A_∞ -sense with the triangles being inherited from the triangles in Ch (where they correspond to the usual cone-construction for chain complexes). There is a Yoneda embedding $\mathcal{Y} : \mathcal{Fuk}^*(M) \rightarrow \text{mod}(\mathcal{Fuk}^*(M))$, the functor associated to an object $L \in \mathcal{L}^*(M)$ being $CF(-, L)$. The derived Fukaya category $D\mathcal{Fuk}^*(M)$ is the homology category associated to the triangulated completion of the image of the Yoneda embedding inside $\text{mod}(\mathcal{Fuk}^*(M))$.

3.1.1. Iterated cone decompositions. We now briefly fix the notation for writing iterated cone-decompositions in a triangulated category \mathcal{C} . Suppose that there are exact triangles:

$$C_{i+1} \rightarrow Z_i \rightarrow Z_{i+1}$$

with $1 \leq i \leq n$ and with $X = Z_{n+1}$, $Z_0 = C_0$. We write such an iterated cone-decomposition as

$$X = (C_{n+1} \rightarrow (C_n \rightarrow (C_{n-1} \rightarrow \dots \rightarrow C_0)) \dots) .$$

With this notation

$$Z_k = (C_k \rightarrow (C_{k-1} \rightarrow \dots \rightarrow C_0)) \dots) .$$

We also notice that we can in fact omit the parentheses in this notation without ambiguity. This follows from the following equality of the two iterated cones:

$$((A \rightarrow B) \rightarrow C) = (A \rightarrow (B \rightarrow C)) .$$

In turn, this follows immediately from the axioms of a triangulated category together with the fact that we work here in an ungraded setting (the formula can also be easily adjusted to the graded case). In short, we will write:

$$X = (C_{n+1} \rightarrow C_n \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_0) .$$

There is a slight abuse of notation in the above formula in that, in the absence of the relevant parentheses, the arrows in the formula do not independently correspond to morphisms in the category \mathcal{C} . The formula should be interpreted as saying that X can be expressed as an iterated cone attachment with the objects C_0, \dots, C_{n+1} as described above.

3.1.2. The Grothendieck group. The Grothendieck group of a triangulated category \mathcal{C} is the abelian group generated by the objects of \mathcal{C} modulo the relations generated by $B = A + C$ as soon as

$$A \rightarrow B \rightarrow C$$

is an exact triangle. We denote the Grothendieck group of \mathcal{C} by $K_0(\mathcal{C})$. Notice that, with our terminology, if

$$L_1 = (L_n \rightarrow L_{n-1} \rightarrow L_{n-2} \rightarrow \dots \rightarrow L_2),$$

then, because we work in an ungraded setting, in $K_0(\mathcal{C})$ we have the relation $L_n + L_{n-1} + \dots + L_1 = 0$. Notice also that, due to the same reason, our version of $K_0(\mathcal{C})$ is always 2-torsion, i.e. $2A = 0$ for every $A \in K_0(\mathcal{C})$.

The main Grothendieck groups of interest in this paper will be those of derived Fukaya categories $K_0 D\mathcal{F}uk^*(-)$.

3.2. Strongly monotone Lefschetz fibrations. In order to define a Fukaya category of cobordisms in a Lefschetz fibration that is suitable for our needs we need to impose additional conditions on the Lefschetz fibration. These will ensure that all the thimbles and vanishing spheres are monotone Lagrangian submanifolds (with the right monotonicity parameters) in their respective ambient manifolds and so can be included as objects in the same Fukaya categories.

Let $\pi : E \rightarrow \mathbb{C}$ be a Lefschetz fibration as in Definition 2.1.1. Fix a base point $z_0 \in \mathbb{C}$ and let $M = \pi^{-1}(z_0)$ be the fiber over z_0 , endowed with the symplectic structure $\omega = \Omega_E|_M$ induced from E . Denote by $x_1, \dots, x_m \in E$ the critical points of π and by $v_1, \dots, v_m \in \mathbb{C}$ the corresponding critical values of π . Fix m smooth paths $\lambda_1, \dots, \lambda_m \subset \mathbb{C}$ such that for every k λ_k starts at v_k and ends at z_0 and such that except of their end points none of the paths λ_k passes through the critical values of π . Denote by $S_1, \dots, S_m \subset M$ the Lagrangian vanishing spheres associated to the paths $\lambda_1, \dots, \lambda_m$.

Definition 3.2.1 (Strongly monotone Lefschetz fibrations). We say that $\pi : E \rightarrow \mathbb{C}$ is a *strongly monotone* Lefschetz fibration if the following conditions holds:

- (1) In case $\dim_{\mathbb{R}} M \geq 4$ we require that M is a monotone symplectic manifold, that is $\omega = 2\rho c_1$ on $\pi_2(M)$ for some $\rho \geq 0$.
- (2) In case $\dim_{\mathbb{R}} M = 2$ we require that (E, Ω_E) is a monotone symplectic manifold. Note that this implies that M is monotone too and we define ρ as in point (1) above.

In addition to the above we also make the following assumptions. Denote by $c_1^{\min} \in \mathbb{Z}_{>0}$ the minimal Chern number of M . Then:

- (i) If $\rho = 0$ set $d_E = 0$ and $*$ = (0).

- (ii) If $\rho > 0$ and $c_1^{\min} = 1$ then we require that $d_{S_1} = \dots = d_{S_m}$ (see Page 20 for what d_{S_k} is). Denote the latter number by $d_E \in \mathbb{Z}_2$. In case $d_E = 0$ set $*$ = (0) and if $d_E = 1$ set $*$ = $(\rho, 1)$.
- (iii) If $c_1^{\min} > 1$ set $d_E = 0$ and $*$ = (0).

We will refer to $*$ from Definition 3.2.1 as the monotonicity class of the Lefschetz fibration E . By Proposition 3.2.3 below it depends only on the fibration E . In §3.3 below we will set up the Fukaya category of (negative ended) cobordisms in E and the monotonicity class $*$ will be used in order to constrain the class of Lagrangian cobordisms that are objects of this category.

We will make one exception to the definition above, namely when E has no critical values at all, i.e. $E \approx \mathbb{C} \times M$ is the trivial fibration. In this case we only assume that M is a monotone symplectic manifold and will choose the monotonicity class $*$ to be arbitrary subject to the restrictions made on page 21 in §3.1 above. See also Remark 4.3.2 below.

Remark 3.2.2. It is easy to see that when $\dim_{\mathbb{R}} M \geq 4$, (M, ω) is monotone iff (E, Ω_E) is monotone and in that case $c_1^{\min}(E) = c_1^{\min}(M)$. This is so because under this dimension assumption, the map induced by inclusion $\pi_2(M) \rightarrow \pi_2(E)$ is surjective. Apart from that we also have $c_1(E)|_{H^2(M)} = c_1(M)$. Moreover, as will be seen in the proof of Proposition 3.2.3 below, the monotonicity of the symplectic manifold (E, Ω_E) implies that the spheres $S_1, \dots, S_k \subset M$ are all monotone (even when $\dim_{\mathbb{R}} M = 2$).

Proposition 3.2.3. *The Definition 3.2.1 is independent of the choice of paths $\lambda_1, \dots, \lambda_m$.*

Let E be a strongly monotone Lefschetz fibration and T a thimble over any path γ (that starts at a critical value of π). Then T is monotone with minimal Maslov number $2c_1^{\min}(E)$ and monotonicity ratio ρ . If moreover, γ is horizontal at $-\infty$ (or $+\infty$) and S is the Lagrangian sphere associated to the end of T then we also have $d_T = d_E = d_S$. In particular, both T and S are monotone of class $$ in their respective ambient manifolds.*

Proof. That all thimbles are monotone follows easily from the fact that T is simply connected and that (E, Ω_E) is a monotone symplectic manifold.

Denote now by T_{λ_k} the thimble over the path λ_k . Since T_{λ_k} is monotone then so is S_k because $c_1(E)|_{H^2(M)} = c_1(M)$.

We now turn to the first statement in the proposition. This follows from the fact that if we change the given set of paths $\lambda_1, \dots, \lambda_m$ by another set $\lambda'_1, \dots, \lambda'_m$ then each of the new vanishing spheres S'_k is the image of S_k under some symplectic diffeomorphism of M (which is in fact, up to symplectic isotopy, a certain composition of Dehn twists and their inverses along the spheres S_1, \dots, S_m). Therefore, the monotonicity of S'_k is preserved and so is the value of $d_{S'_k}$.

Finally, let T be a thimble over a path γ which is horizontal at $\pm\infty$. By the results of [Che] (see also [BC2, Remark 2.2.4]) with obvious adaption to Lefschetz fibrations it follows that $d_T = d_S$, where S is the Lagrangian sphere associated to the end of T . Since S is a vanishing sphere we have $d_S = d_E$. \square

Remark 3.2.4. The procedure from Proposition 2.3.1, that modifies the symplectic structure on a Lefschetz fibration to render it tame, does not affect the property of being strongly monotone. This is so because, in the notation of Proposition 2.3.1, the map induced by the inclusion $\pi_2(\pi^{-1}(\mathcal{N})) \rightarrow \pi_2(E)$ is an isomorphism.

From now on, we will generally assume that our Lefschetz fibrations are strongly monotone.

3.3. The Fukaya category of negative ended cobordisms in tame Lefschetz fibrations. We consider a strongly monotone Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$ that is tame outside $U \subset \mathbb{C}$ and has as generic fibre the symplectic manifold (M, ω) . We will also assume that U is U -shaped, as in Figure 6, and that

$$(18) \quad \overline{U} \subset \mathbb{R} \times [0, +\infty).$$

The main object of study in this paper is the Fukaya category $\mathcal{Fuk}^*(E)$, where $*$ is the monotonicity class of E and has been set in Definition 3.2.1. It has as objects the cobordisms V as in Definition 2.2.3 such that the following additional conditions are satisfied:

- i. V is monotone in the class $*$.
- ii. $V \subset \pi^{-1}(\mathbb{R} \times [\frac{1}{2}, +\infty))$
- iii. V has only negative ends that all belong to $\mathcal{L}^*(M)$. In particular, with the notation from Definition 2.2.3, $k_+ = 1$ and $L'_1 = \emptyset$.

This family of Lagrangians of E with the properties above will be denoted by $\mathcal{L}^*(E)$. In other words, $\mathcal{Ob}(\mathcal{Fuk}^*(E)) = \mathcal{L}^*(E)$. Such an object is represented schematically in Figure 6.

We call the objects $V \in \mathcal{L}^*(E)$ *negatively-ended cobordisms*: they are cobordisms from the void set to a family (L_1, \dots, L_s) .

Remark 3.3.1. a. In this paper we restrict ourselves to negatively-ended cobordisms but this is more a matter of convenience than of necessity. Some of the arguments in the paper are simpler in this setting but the same type of constructions allow the definition of a Fukaya category with both negative and positive ends. Similarly, our decomposition results can also be adapted to this more general setting. We do not require V to be connected. Notice also that every Lagrangian cobordism $V \subset E$ that contains positive ends can be transformed to a negatively-ended cobordism by e.g. bending its positive ends along curves that turn to the left, then go above the singularities of E and continue horizontally to $-\infty$.

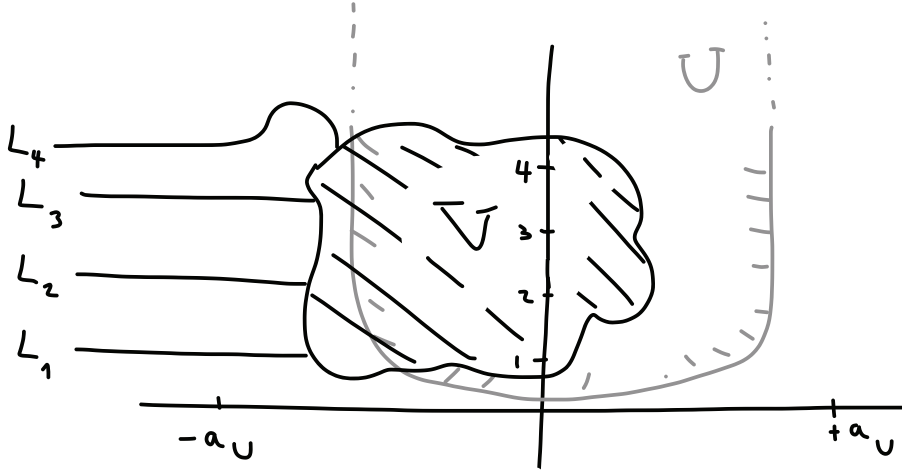


FIGURE 6. The projection on \mathbb{C} of an object $V \in \mathcal{Ob}(\mathcal{Fuk}^*(E))$ together with the set U outside which E is tame.

- b. We remark that our notation $\mathcal{L}^*(E)$ and $\mathcal{Fuk}^*(E)$ somewhat differs from the one used in [BC3]. In that paper we studied Lagrangian cobordisms in trivial fibrations $E = \mathbb{C} \times M$ and denoted by $\mathcal{CL}_d(\mathbb{C} \times M)$ the collection of monotone Lagrangian cobordisms in $\mathbb{C} \times M$ (with possibly negative and positive ends). The corresponding Fukaya category was denoted by $\mathcal{Fuk}_{cob}^d(\mathbb{C} \times M)$. Thus, in the present paper, we could have denoted our $\mathcal{L}^*(E)$ by $\mathcal{CL}_*^{null}(E)$ and $\mathcal{Fuk}^*(E)$ by $\mathcal{Fuk}_{cob}^{*,null}(E)$, but we have decided to drop the additional decorations in order to keep the notation simpler.

The operations μ_k of the Fukaya category $\mathcal{Fuk}^*(E)$ are defined following closely the construction in [BC3] which is basically a variant of the set-up in Seidel's book [Sei3, Sections 8-12]. We review here the technical points that will be needed later in the paper. We will first focus on the case when M is compact and we will discuss the additional modifications required when M is convex at infinity at the end of the construction. There are two structures that need to be added compared to the construction of the category $\mathcal{Fuk}^*(M)$: *transition functions* associated to a system of strip-like ends and *profile functions*. As always, the operations μ_k are defined in terms of counting (with coefficients in \mathcal{A}) perturbed J -holomorphic polygons u . The role of the transition functions is to allow such u to be transformed by a change of variables into curves v that project holomorphically onto certain regions of \mathbb{C} . The role of the profile functions - and particularly that of their *bottlenecks* - is to ensure compactness at infinity for the Floer complexes $CF(V, V')$ and to further restrict the behavior of the J -polygons u . We explain this point, which is crucial for the arguments used later in the paper, at the end of §3.3.

3.3.1. *Transition functions.* We first recall the notion of a consistent choice of strip-like ends from [Sei3, Sections 8d, 9g]. Fix $k \geq 2$. Let $\text{Conf}_{k+1}(\partial D)$ be the space of configurations of $(k+1)$ distinct points (z_1, \dots, z_{k+1}) on ∂D that are ordered clockwise. Denote by $\text{Aut}(D) \cong \text{PLS}(2, \mathbb{R})$ the group of holomorphic automorphisms of the disk D . Let

$$\mathcal{R}^{k+1} = \text{Conf}_{k+1}(\partial D) / \text{Aut}(D), \quad \widehat{\mathcal{S}}^{k+1} = (\text{Conf}_{k+1}(\partial D) \times D) / \text{Aut}(D).$$

The projection $\widehat{\mathcal{S}}^{k+1} \rightarrow \mathcal{R}^{k+1}$ has sections $\zeta_i[z_1, \dots, z_{k+1}] = [(z_1, \dots, z_{k+1}), z_i]$, $i = 1, \dots, k+1$ and let $\mathcal{S}^{k+1} = \widehat{\mathcal{S}}^{k+1} \setminus \bigcup_{i=1}^{k+1} \zeta_i(\mathcal{R}^{k+1})$. The fiber bundle $\mathcal{S}^{k+1} \rightarrow \mathcal{R}^{k+1}$ is called a universal family of $(k+1)$ -pointed disks. Its fibers S_r , $r \in \mathcal{R}^{k+1}$, are called $(k+1)$ -pointed (or punctured) disks.

Let $Z^+ = [0, \infty) \times [0, 1]$, $Z^- = (-\infty, 0] \times [0, 1]$ be the two infinite semi-strips and let S be a $(k+1)$ pointed disk with punctures at (z_1, \dots, z_{k+1}) . A choice of strip-like ends for S is a collection of embeddings: $\epsilon_i^S : Z^- \rightarrow S$, $1 \leq i \leq k$, $\epsilon_{k+1}^S : Z^+ \rightarrow S$ that are proper and holomorphic and

$$\begin{aligned} (\epsilon_i^S)^{-1}(\partial S) &= (-\infty, 0] \times \{0, 1\}, & \lim_{s \rightarrow -\infty} \epsilon_i^S(s, t) &= z_i, \quad \forall 1 \leq i \leq k, \\ (\epsilon_{k+1}^S)^{-1}(\partial S) &= [0, \infty) \times \{0, 1\}, & \lim_{s \rightarrow \infty} \epsilon_{k+1}^S(s, t) &= z_{k+1}. \end{aligned}$$

such that the ϵ_i^S 's have pairwise disjoint images. A universal choice of strip-like ends for $\mathcal{S}^{k+1} \rightarrow \mathcal{R}^{k+1}$ is a choice of $k+1$ proper embeddings $\epsilon_i^S : \mathcal{R}^{k+1} \times Z^- \rightarrow \mathcal{S}^{k+1}$, $i = 1, \dots, k$, $\epsilon_{k+1}^S : \mathcal{R}^{k+1} \times Z^+ \rightarrow \mathcal{S}^{k+1}$ such that for every $r \in \mathcal{R}^{k+1}$ the restrictions $\epsilon_i^S|_{r \times Z^\pm}$ consists of a choice of strip-like ends for S_r . See [Sei3, Section 9c] for more details. In the case $k = 1$, we put $\mathcal{R}^2 = \text{pt}$ and $\mathcal{S}^2 = D \setminus \{-1, 1\}$. We endow $D \setminus \{-1, 1\}$ with strip-like ends by identifying it holomorphically with the strip $\mathbb{R} \times [0, 1]$, where the latter is endowed with its standard complex structure. The identification is done such that $-1 \in D$ corresponds to $-\infty \times [0, 1]$ and $+1 \in D$ to $+\infty \times [0, 1]$.

Pointed disks with strip-like ends can be glued in a natural way. Further, the space \mathcal{R}^{k+1} has a natural compactification $\overline{\mathcal{R}}^{k+1}$ described by parametrizing the elements of $\overline{\mathcal{R}}^{k+1} \setminus \mathcal{R}^{k+1}$ by trees [Sei3]. The family $\mathcal{S}^{k+1} \rightarrow \mathcal{R}^{k+1}$ admits a partial compactification $\overline{\mathcal{S}}^{k+1} \rightarrow \overline{\mathcal{R}}^{k+1}$ which can be endowed with a smooth structure. Moreover, the fixed choice of universal strip-like ends for $\mathcal{S}^{k+1} \rightarrow \mathcal{R}^{k+1}$ admits an extension to $\overline{\mathcal{S}}^{k+1} \rightarrow \overline{\mathcal{R}}^{k+1}$. Further, these choices of universal strip-like ends for the spaces \mathcal{R}^{k+1} for different k 's can be made in a way consistent with these compactifications (see [Sei3, Sections 9d, 9e] and Lemma 9.3 in that book).

Our construction requires the additional auxiliary structure of *transition functions*. This structure can be defined once a choice of universal strip-like ends is fixed. It consists of a smooth function $\mathbf{a}^{k+1} : \mathcal{S}^{k+1} \rightarrow [0, 1]$ with the following properties. First let $k = 1$. In this case $\mathcal{S}^2 = D \setminus \{-1, 1\} \cong \mathbb{R} \times [0, 1]$ and we define $\mathbf{a}^2(s, t) = t$, where $(s, t) \in \mathbb{R} \times [0, 1]$. To

describe \mathbf{a}^{k+1} for $k \geq 2$ write $a_r := \mathbf{a}^{k+1}|_{S_r}$, $r \in \mathcal{R}^{k+1}$. We require the functions a_r to satisfy the following for every $r \in \mathcal{R}^{k+1}$ - see Figure 7:

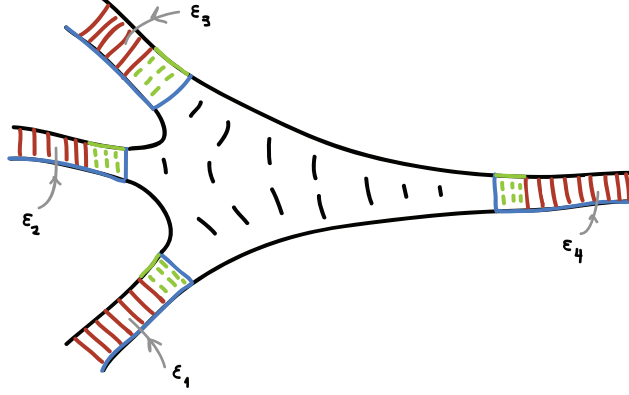


FIGURE 7. The constraints imposed on a transition function for a domain with three entries and one exit: in the red region the function a equals $(s, t) \rightarrow t$; along the blue arcs the function a vanishes; the green region is a transition region. There are no additional constraints in the black region.

- i. For each entry strip-like end $\epsilon_i : Z^- \rightarrow S_r$, $1 \leq i \leq k$, we have:
 - a. $a_r \circ \epsilon_i(s, t) = t$, $\forall (s, t) \in (-\infty, -1] \times [0, 1]$.
 - b. $\frac{\partial}{\partial s}(a_r \circ \epsilon_i)(s, 1) \leq 0$ for $s \in [-1, 0]$.
 - c. $a_r \circ \epsilon_i(s, t) = 0$ for $(s, t) \in ((-\infty, 0] \times \{0\}) \cup (\{0\} \times [0, 1])$.
- ii. For the exit strip-like end $\epsilon_{k+1} : Z^+ \rightarrow S_r$ we have:
 - a'. $a_r \circ \epsilon_{k+1}(s, t) = t$, $\forall (s, t) \in [1, \infty) \times [0, 1]$.
 - b'. $\frac{\partial}{\partial s}(a_r \circ \epsilon_{k+1})(s, 1) \geq 0$ for $s \in [0, 1]$.
 - c'. $a_r \circ \epsilon_{k+1}(s, t) = 0$ for $(s, t) \in ([0, +\infty) \times \{0\}) \cup (\{0\} \times [0, 1])$.

The total function $\mathbf{a}^{k+1} : \mathcal{S}^{k+1} \rightarrow [0, 1]$ will be called a global transition function. The functions \mathbf{a}^{k+1} can be picked consistently for different values of k in the sense that \mathbf{a} extends smoothly to $\bar{\mathcal{S}}^{k+1}$ and along the boundary $\partial \bar{\mathcal{S}}^{k+1}$ it coincides with the corresponding pairs of functions $\mathbf{a}^{k'+1} : \mathcal{S}^{k'+1} \rightarrow [0, 1]$, $\mathbf{a}^{k''+1} : \mathcal{S}^{k''+1} \rightarrow [0, 1]$ with $k' + k'' = k + 1$, associated to trees of split pointed disks.

3.3.2. Profile function. We now discuss the second special ingredient in our construction: profile functions.

To fix ideas we suppose from now on in this construction that

$$(19) \quad U \subset \left[-\frac{1}{2}, \frac{1}{2}\right] \times [0, \infty).$$

According to the notation in (4) and together with (18) this means that $a_U \leq \frac{1}{2}$. (The real number a_U from (4) should not be confused with the functions a_r from the preceding section.)

We will use a *profile function*: $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ which, by definition, has the following properties (see Figure 8):

- i. The support of h is contained in the union of the sets

$$W_i^+ = [2, \infty) \times [i - \epsilon, i + \epsilon] \quad \text{and} \quad W_i^- = (-\infty, -1] \times [i - \epsilon, i + \epsilon], \quad i \in \mathbb{Z},$$

where $0 < \epsilon < 1/4$.

- ii. The restriction of h to each set $F_i^+ = [2, \infty) \times [i - \epsilon/2, i + \epsilon/2]$ and $F_i^- = (-\infty, -1] \times [i - \epsilon/2, i + \epsilon/2]$ is respectively of the form $h(x, y) = h_{\pm}(x)$, where the smooth functions h_{\pm} satisfy:
 - a. $h_- : (-\infty, -1] \rightarrow \mathbb{R}$ has a single critical point in $(-\infty, -1]$ at $-\frac{3}{2}$ and this point is a non-degenerate local maximum. Moreover, for all $x \in (-\infty, -2)$, we have $h_-(x) = \alpha^- x + \beta^-$ for some constants $\alpha^-, \beta^- \in \mathbb{R}$ with $\alpha^- > 0$.
 - b. $h_+ : [2, \infty) \rightarrow \mathbb{R}$ has a single critical point in $[2, \infty)$ at $\frac{5}{2}$ and this point is also a non-degenerate maximum. Moreover, for all $x \in (3, \infty)$ we have $h_+(x) = \alpha^+ x + \beta^+$ for some constants $\alpha^+, \beta^+ \in \mathbb{R}$ with $\alpha^+ < 0$.
- iii. The Hamiltonian isotopy $\phi_t^h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ associated to h exists for all $t \in \mathbb{R}$; the derivatives of the functions h_{\pm} are sufficiently small such that the Hamiltonian isotopy ϕ_t^h keeps the sets $[2, \infty) \times \{i\}$ and $(-\infty, -1] \times \{i\}$ inside the respective F_i^{\pm} for $-1 \leq t \leq 1$.
- iv. The Hamiltonian isotopy ϕ_t^h preserves the strip $[-\frac{3}{2}, \frac{5}{2}] \times \mathbb{R}$ for all t , in other words $\phi_t^h([-\frac{3}{2}, \frac{5}{2}] \times \mathbb{R}) = [-\frac{3}{2}, \frac{5}{2}] \times \mathbb{R}$ for every t .

Such functions h are easy to construct. Their main role is to disjoin the ends corresponding to two (or more) cobordisms at $\pm\infty$. The critical points $(-3/2, i)$ and $(5/2, i)$ are called *bottlenecks*.

3.3.3. Perturbation data, J -holomorphic polygons and μ_k . At this step we describe the (perturbed) J -holomorphic polygons that define the μ_k 's.

The construction of μ_k starts with μ_1 and the so-called Floer datum. For each pair of cobordisms $V, V' \subset E$ the Floer datum $\mathcal{D}_{V, V'} = (\bar{H}_{V, V'}, J_{V, V'})$ consists of a Hamiltonian $\bar{H}_{V, V'} : [0, 1] \times E \rightarrow \mathbb{R}$ and a (possibly time dependent) almost complex structure $J_{V, V'}$ on E which is compatible with Ω_E . We will also assume that each Floer datum $(\bar{H}_{V, V'}, J_{V, V'})$ satisfies the following conditions:

- i. $\phi_1^{\bar{H}_{V, V'}}(V)$ is transverse to V' .

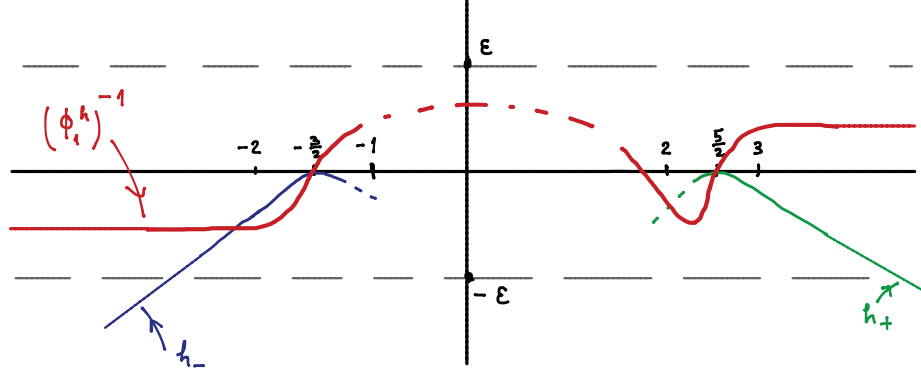


FIGURE 8. The graphs of h_- and h_+ and the image of \mathbb{R} by the Hamiltonian diffeomorphism $(\phi_1^h)^{-1}$. The profile of the functions h_- at $-3/2$ and h_+ at $5/2$ are the “bottlenecks”.

- ii. Write points of $E \setminus \pi^{-1}(U)$ as (x, y, p) with $x + iy \in \mathbb{C}$, $p \in M$. We require that there exists a compact set $K_{V,V'} \subset (-\frac{5}{4}, \frac{9}{4}) \times \mathbb{R} \subset \mathbb{C}$ such that $\bar{H}_{V,V'}(t, (x, y, p)) = h(x, y) + H_{V,V'}(t, p)$ for $(x + iy, p)$ outside of $\pi^{-1}(K_{V,V'})$, for some $H_{V,V'} : [0, 1] \times M \rightarrow \mathbb{R}$.
- iii. The projection $\pi : E \rightarrow \mathbb{C}$ is $(J_{V,V'}(t), (\phi_t^h)_* i)$ -holomorphic outside of $\pi^{-1}(K_{V,V'})$ for every $t \in [0, 1]$.

Remark 3.3.2. The almost complex structure $J_{V,V'}$ can be viewed in some sense as a perturbation of the almost complex structure J_E that is part of the Lefschetz fibration structure as in Definition 2.1.1. Indeed, if the profile function h is taken to be arbitrarily small then $J_{V,V'}$ can be chosen to be arbitrarily close to J_E . In practice we will not take this viewpoint and will not insist that $J_{V,V'}$ is a good approximation of J_E .

The time-1 Hamiltonian chords $\mathcal{P}_{\bar{H}_{V,V'}}$ of $\bar{H}_{V,V'}$ that start on V and end on V' , form a finite set.

For a $(k + 1)$ -pointed disk S_r , let $C_i \subset \partial S_r$ be the connected components of ∂S_r indexed so that C_1 goes from the exit to the first entry, C_i goes from the $(i - 1)$ -th entry to the i , $1 \leq i \leq k$, and C_{k+1} goes from the k -th entry to the exit.

Following Seidel’s scheme from [Sei3, Section 9], we now need to choose additional perturbation data.

For every collection of cobordisms V_i , $1 \leq i \leq k + 1$ we choose a perturbation datum $\mathcal{D}_{V_1, \dots, V_{k+1}} = (\Theta, \mathbf{J})$ consisting of:

- I. A family $\Theta = \{\Theta^r\}_{r \in \mathcal{R}^{k+1}}$, where $\Theta^r \in \Omega^1(S_r, C^\infty(E))$ is a 1-form on S_r with values in smooth functions on E . We write $\Theta^r(\xi) : E \rightarrow \mathbb{R}$ for the value of Θ^r on $\xi \in TS_r$.
- II. $\mathbf{J} = \{J_z\}_{z \in S^{k+1}}$ is a family of Ω_E -compatible almost complex structure on E , parametrized by $z \in S_r$, $r \in \mathcal{R}^{k+1}$.

The forms Θ^r induce forms $Y^r = Y^{\Theta^r} \in \Omega^1(S_r, C^\infty(TE))$ with values in (Hamiltonian) vector fields on E via the relation $Y(\xi) = X^{\Theta(\xi)}$ for each $\xi \in TS_r$ (i.e. $Y(\xi)$ is the Hamiltonian vector field on E associated to the autonomous Hamiltonian function $\Theta(\xi) : E \rightarrow \mathbb{R}$).

The relevant Cauchy-Riemann equation associated to $\mathcal{D}_{V_1, \dots, V_{k+1}}$ is:

$$(20) \quad u : S_r \rightarrow E, \quad Du + J(z, u) \circ Du \circ j = Y + J(z, u) \circ Y \circ j, \quad u(C_i) \subset V_i.$$

Here j stands for the complex structure on S_r . The i -th entry of S_r is labeled by a time-1 Hamiltonian orbit $\gamma_i \in \mathcal{P}_{\bar{H}_{V_i, V_{i+1}}}$ and the exit is labeled by a time-1 Hamiltonian orbit $\gamma_{k+1} \in \mathcal{P}_{\bar{H}_{V_1, V_{k+1}}}$. The map u satisfies $u(C_i) \subset V_i$ and u is required to be asymptotic - in the usual Floer sense - to the Hamiltonian orbits γ_i on each respective strip-like end. See [Sei3, Section 8f] for more details on this equation, the boundary conditions and the asymptotics.

The perturbation data $\mathcal{D}_{V_1, \dots, V_{k+1}}$ are constrained by a number of additional conditions that we now describe. First, denote by $s_{V_1, \dots, V_{k+1}} \in \mathbb{N}$ the smallest $l \in \mathbb{N}$ such that $\pi(V_1 \cup \dots \cup V_{k+1}) \subset \mathbb{R} \times (0, l)$. Write $\bar{h} = h \circ \pi : E \rightarrow \mathbb{R}$, where $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the profile function fixed before. We also write

$$\begin{aligned} U_i^r &= \epsilon_i^{S_r}((-\infty, -1] \times [0, 1]) \subset S_r, \quad i = 1, \dots, k, \\ U_{k+1}^r &= \epsilon_{k+1}^{S_r}([1, \infty) \times [0, 1]) \subset S_r, \\ \mathcal{W}^r &= \bigcup_{i=1}^{k+1} U_i^r. \end{aligned}$$

The conditions on $\mathcal{D}_{V_1, \dots, V_{k+1}}$ are the following:

- a. *Asymptotic conditions.* For every $r \in \mathcal{R}^{k+1}$ we have $\Theta|_{U_i^r} = \bar{H}_{V_i, V_{i+1}} dt$, $i = 1, \dots, k$ and $\Theta|_{U_{k+1}^r} = \bar{H}_{V_1, V_{k+1}} dt$. (Here (s, t) are the coordinates parametrizing the strip-like ends.) Moreover, on each U_i^r , $i = 1, \dots, k$, J_z coincides with $J_{V_i, V_{i+1}}$ and on U_{k+1}^r it coincides with $J_{V_1, V_{k+1}}$, i.e. $J_{\epsilon_i^{S_r}(s, t)} = J_{V_i, V_{i+1}}(t)$ and similarly for the exit end. Thus, over the part of the strip-like ends \mathcal{W}^r the perturbation datum $\mathcal{D}_{V_1, \dots, V_{k+1}}$ is compatible with the Floer data $\mathcal{D}_{V_i, V_{i+1}}$, $i = 1, \dots, k$ and $\mathcal{D}_{V_1, V_{k+1}}$.
- b. *Special expression for Θ .* The restriction of Θ to S_r equals

$$\Theta|_{S_r} = da_r \otimes \bar{h} + \Theta_0$$

for some $\Theta_0 \in \Omega^1(S_r, C^\infty(E))$ which depends smoothly on $r \in \mathcal{R}^{k+1}$. Here $a_r : S_r \rightarrow \mathbb{R}$ are the transition functions fixes at the point 1. The form Θ_0 is required to satisfy the following two conditions:

- 1. $\Theta_0(\xi) = 0$ for all $\xi \in TC_i \subset T\partial S_r$.

2. There exists a compact set $K_{V_1, \dots, V_{k+1}} \subset (-\frac{3}{2}, \frac{5}{2}) \times \mathbb{R}$ which is independent of $r \in \mathcal{R}^{k+1}$ such that $\pi^{-1}(K_{V_1, \dots, V_{k+1}})$ contains all the sets K_{V_i, V_j} involved in the Floer datum \mathcal{D}_{V_i, V_j} , and with

$$K_{V_1, \dots, V_{k+1}} \supset \left(\left[-\frac{5}{4}, \frac{9}{4} \right] \times [-s_{V_1, \dots, V_{k+1}}, +s_{V_1, \dots, V_{k+1}}] \right)$$

such that outside of $\pi^{-1}(K_{V_1, \dots, V_{k+1}})$ we have $D\pi(Y_0) = 0$ for every r , where $Y_0 = X^{\Theta_0}$.

- c. Outside of $\pi^{-1}(K_{V_1, \dots, V_{k+1}})$ the almost complex structure \mathbf{J} has the property that the projection π is $(J_z, (\phi_{a_r(z)}^h)_*(i))$ -holomorphic for every $r \in \mathcal{R}^{k+1}$, $z \in S_r$.

Using the above choices of data we construct the A_∞ -category $\mathcal{Fuk}^*(E)$ by the construction from [Sei3, Section 9] with the modifications described in [BC3] that are needed due to the fact that the Lagrangians are not compact. As mentioned before, the objects of this category are Lagrangians cobordisms $V \subset E$ without positive ends that are uniformly monotone of class $*$, the morphisms space between the objects V and V' are $CF(V, V'; \mathcal{D}_{V, V'})$, the \mathcal{A} -vector space generated by the Hamiltonian chords $\mathcal{P}_{\bar{H}_{V, V'}}$. The A_∞ structural maps

$$\mu_k : CF(V_1, V_2) \otimes CF(V_2, V_3) \otimes \dots \otimes CF(V_k, V_{k+1}) \rightarrow CF(V_1, V_{k+1})$$

are defined by summing - with coefficients in \mathcal{A} - pairs (r, u) with $r \in \mathcal{R}^{k+1}$ and u a finite energy solution of (20) that belongs to a 0-dimensional moduli space. The coefficient in front of a perturbed J -holomorphic polygon u is $T^{\omega(u)}$. The Gromov compactness and regularity arguments work just as in [BC3]. (The fact that in that paper the total space was $E = \mathbb{C} \times M$ whereas here E is a Lefschetz fibration plays no role in these arguments.) In fact, as we work here over the universal Novikov ring compactness is easier to establish in this case (and we do not require the vanishing of the inclusions $\pi_1(V) \rightarrow \pi_1(E)$ as in [BC3]).

The choice of strip-like ends, transition functions and profile function (in particular, the placement of the bottlenecks) changes the resulting A_∞ -category only up to quasi-equivalence.

Once the category $\mathcal{Fuk}^*(E)$ is constructed the derived category $D\mathcal{Fuk}^*(E)$ is defined by again considering the A_∞ -modules $mod(\mathcal{Fuk}^*(M)) := fun(\mathcal{Fuk}^*(E), Ch^{opp})$ and by letting $D\mathcal{Fuk}^*(E)$ be the homological category associated to the triangulated closure of the image of the Yoneda functor $\mathcal{Y} : \mathcal{Fuk}^*(E) \rightarrow mod(\mathcal{Fuk}^*(E))$.

3.3.4. The naturality transformation. Assume that $u : S_r \rightarrow E$ is a solution of (20), where the Floer and perturbation data satisfy the conditions discussed at the points a , b , c on page 32. Define $v : S_r \rightarrow E$ by the formula:

$$(21) \quad u(z) = \phi_{a_r(z)}^{\bar{h}}(v(z)),$$

where $a_r : S_r \rightarrow [0, 1]$ is the transition function.

The Floer equation (20) for u transforms into the following equation for v :

$$(22) \quad Dv + J'(z, v) \circ Dv \circ j = Y' + J'(z, v) \circ Y' \circ j.$$

Here $Y' \in \Omega^1(S_r, C^\infty(TM))$ and J' are defined by:

$$(23) \quad Y = D\phi_{a(z)}^{\bar{h}}(Y') + da_r \otimes X^{\bar{h}}, \quad J_z = (\phi_{a_r(z)}^{\bar{h}})_* J'_z.$$

The map v satisfies the following moving boundary conditions:

$$(24) \quad \forall z \in C_i, \quad v(z) \in (\phi_{a(z)}^{\bar{h}})^{-1}(V_i).$$

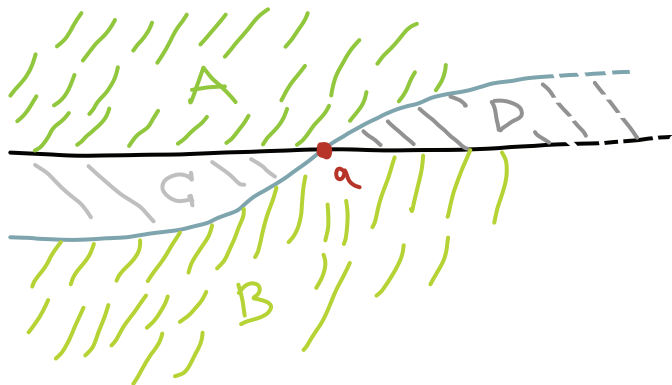
The asymptotic conditions for v at the punctures of S_r are as follows. For $i = 1, \dots, k$, $v(\epsilon_i(s, t))$ tends as $s \rightarrow -\infty$ to a time-1 chord of the flow $(\phi_t^{\bar{h}})^{-1} \circ \phi_t^{\bar{H}_{V_i, V_{i+1}}}$ starting on V_i and ending on $(\phi_1^{\bar{h}})^{-1}(V_{i+1})$. (Here $\epsilon_i(s, t)$ is the parametrization of the strip-like end at the i 'th puncture.) Similarly, $v(\epsilon_{k+1}(s, t))$ tends as $s \rightarrow \infty$ to a chord of $(\phi_t^{\bar{h}})^{-1} \circ \phi_t^{\bar{H}_{V_1, V_{k+1}}}$ starting on V_1 and ending on $(\phi_1^{\bar{h}})^{-1}(V_{k+1})$.

It might be useful to spell out more geometrically the effect of the moving boundary conditions (24) on the ends of the Lagrangians V_i . Identify a neighborhood of puncture number i , $1 \leq i \leq k$, in S_r with $Z^- = (-\infty, 0] \times [0, 1]$ via the strip-like ends construction as in §3.3.1. Then for every $x \in (-\infty, 0]$, we have $v(x, 0) \in V_i$ and $v(x, 1) \in (\phi_{\alpha(x)}^{\bar{h}})^{-1}(V_{i+1})$, where $\alpha : (-\infty, 0] \rightarrow [0, 1]$ is a function that equals 1 on $(-\infty, -1]$ and on the interval $[-1, 0]$ it decreases from 1 to 0. Note that the part of $(\phi_{\alpha(x)}^{\bar{h}})^{-1}(V_{i+1})$ that lies over $(-\infty, -2] \times \mathbb{R}$ is just $(\phi_1^{\bar{h}})^{-1}(\text{ends of } V_{i+1})$ hence coincides with the ends of V_{i+1} after being pushed downwards (in the y -direction of the \mathbb{C} -factor) by a small amount. See the left-hand side of Figure 8. Note also that for each $s \in \mathbb{N}$ such that both V_i and V_{i+1} have an s -end, i.e. an end over $(-\infty, -a_U] \times \{s\}$, the following happens: the projections $\pi(s\text{-end of } V_i)$ and $\pi((\phi_{\alpha(x)}^{\bar{h}})^{-1}(s\text{-end of } V_{i+1}))$ intersect transversely at the points $(-\frac{3}{2}, s)$. See again Figure 8. A similar description holds also for the exit strip-like end Z^+ .

Let now $v' = \pi \circ v : S_r \rightarrow \mathbb{C}$. It is then easy to see - as in [BC3, Page 1766] - that v' is holomorphic over $\mathbb{C} \setminus ([-\frac{3}{2} + \delta', \frac{5}{2} - \delta'] \times \mathbb{R})$ for small enough $\delta' > 0$.

As discussed in [BC3], there are many useful consequences of the holomorphicity of v' around a bottleneck and we will see some more later in this paper. To give a typical simple example, assume that the bottleneck in question is $a = (-\frac{3}{2}, 0)$ and that the regions A and B in Figure 9 are unbounded. In this case, the image of v' can not switch from region D to region C (or vice-versa). More precisely, it is impossible to have that $\text{Image}(v') \cap C \neq \emptyset$ and $\text{Image}(v') \cap D \neq \emptyset$ with the regions C, D as in the picture.

The argument is as follows: assume that $\text{Image}(v')$ intersects both C and D and is disjoint from the interiors of both A and B . Let $x_1 \in \text{Image}(v') \cap C$ and $x_2 \in \text{Image}(v') \cap D$. Let c be a curve inside the domain of v' that connects x_1 to x_2 . It follows that $a \in v'(c)$. But as

FIGURE 9. The bottleneck a and the regions A , B , C and D .

there are infinitely many distinct curves c joining x_1 to x_2 this means that there are infinitely many interior points z with $v'(z) = a$. But this implies $\text{Image}(v') = a$. Thus $\text{Image}(v')$ has to intersect at least one of A and B and, by the open mapping theorem, this contradicts the fact that the closure of $\text{Image}(v')$ is compact.

This argument is used in several instances in [BC3], for example to show the compactness of the moduli spaces required to define μ_k as well as those used to show $\mu \circ \mu = 0$.

Besides this compactness implication, the holomorphicity of v' has an important role in the proof of the main decomposition result in [BC3] as well as in the main result of the current paper. Both these results are consequences of writing certain A_∞ -module structures μ_k in an “upper triangular” form. In turn, this form is deduced from the fact that the planar projections of the J -holomorphic polygons giving the module multiplications are holomorphic (over an appropriate region in \mathbb{C}) and a “bottleneck-type” argument is used repeatedly to show the vanishing of the relevant components of the μ_k ’s. See for example [BC3, Sections 4.2, 4.4].

3.3.5. The case of a non-compact fibre. We now assume that (M, ω) is non-compact and convex at infinity and that the Lefschetz fibration E satisfies the conditions in §2.1 as well as the Assumption T_∞ from page 9. Additionally, we continue to assume that E is tame outside a U -shaped subset $U \subset \mathbb{C}$ as in §3.3.

From Assumption T_∞ we deduce that there is a trivialization $\phi : \mathbb{C} \times M^\infty \rightarrow E^\infty$ with respect to which both the symplectic form and the almost complex structure split so that, in particular, $\phi^* J_E = j \oplus J_0$ where J_0 is a fixed almost complex structure on M compatible with ω and with the symplectic convexity of M . Recall also that $E^0 = E \setminus E^\infty$.

The objects of the category $\mathcal{Fuk}^*(E)$ are the same as before. Notice that, by Definition 2.2.3, any cobordism V has the property that $V \cap \pi^{-1}(z)$ is compact for any $z \in \mathbb{C}$. Furthermore, all the construction of the category $\mathcal{Fuk}^*(E)$ proceeds exactly in the same fashion as in the compact case with an additional requirement: all the almost complex structures involved are

required to coincide with J_E outside a large enough neighborhood of E^0 . More precisely, for any two objects $V, V' \in \mathcal{Ob}(\mathcal{Fuk}^*(E))$ we require that $J_{V,V'}$ coincide with J_E outside a neighborhood of E^0 that contains both V and V' . Similarly, each almost complex structure J_z in the family \mathbf{J} that is part of the perturbation data associated to the collection of cobordisms V_1, \dots, V_{k+1} has to coincide with J_E outside of a neighborhood of E^0 that contains all of the V_i 's.

Finally, notice that as explained in §3.3.4 the actual curves u that appear in the μ_k 's are transformed into curves v which satisfy equations that are holomorphic with respect to almost complex structures of the form $J'_z = (\phi_{a_r(z)}^{\bar{h}})^{-1} J_z$. Due to the splitting provided by the trivialization ϕ and because $\bar{h} = h \circ \pi$ these structures are also split at ∞ (along the fibre) and, by using the trivialization ϕ , it follows that J'_z restricted to the fiber direction coincides with J_0 (away from a compact subset). Therefore, over E^∞ one can again use ϕ to project such a curve v on M^∞ thus getting a new curve v' that way from a compact is J_0 -holomorphic. The usual compactness arguments for manifolds that are symplectically convex at infinity apply to this v' and thus compactness is achieved without issues.

Remark 3.3.3. In [Sei3] (see also [Sei4]) Seidel introduced a Fukaya category associated to a Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$. By neglecting for a moment some technical points that will be revisited below, the relation between this category and the category $\mathcal{Fuk}^*(E)$ introduced above is that Seidel's category is quasi-equivalent to the subcategory of $\mathcal{Fuk}^*(E)$ with objects the thimbles T_i covering the curves t_i in Figure 1. The technical points are that, firstly, we work in a monotone and ungraded setting and Seidel's work is in the exact and graded case (and the grading plays an important role in his work). Secondly, the type of perturbations at infinity that Seidel uses - see in particular [Sei4] - are different from ours. Despite these differences, it is possible to show that Seidel's approach can also be implemented in the monotone case and the resulting category is quasi-equivalent to the subcategory of $\mathcal{Fuk}^*(E)$ as mentioned above. One reason for not pursuing this direction in this paper is that in the construction of $\mathcal{Fuk}^*(E)$ above we use the perturbations employing bottlenecks etc. These are very convenient if one uses the naturality transformation - as explained in §3.3.4 - to reduce key steps of the proofs in this paper (as well as in [BC3]) to properties of holomorphic planar curves.

3.4. Fukaya categories of negative ended cobordisms in general Lefschetz fibrations. In this section we use the construction in §3.3 to associate a Fukaya A_∞ -category to a general Lefschetz fibration. Let $\pi : E \rightarrow \mathbb{C}$ be a Lefschetz fibration as in §2.1. The category we intend to construct will depend on a tame Lefschetz fibration $\pi : E_\tau \rightarrow \mathbb{C}$ associated to E and will be denoted by $\mathcal{Fuk}^*(E; \tau)$. The parameter τ indicates the choice of a tame symplectic structure on E with the properties described in the construction below.

We first fix an additional notation. For two constants $r < 0 < s$, put $S_{r,s} = [r, s] \times \mathbb{R} \subset \mathbb{C}$. Fix constants $x < 0 < y$ such that all the singularities of the fibration E are contained in the interior of $\pi^{-1}(S_{x,y})$. We also assume that the critical values of π are included in the upper half plane.

The construction is now the following. The objects of the category $\mathcal{Fuk}^*(E; \tau)$ are cobordisms V in E - in the sense of Definition 2.2.1 - that are cylindrical outside $S_{x-3, y+3}$ and satisfy the following additional constraints:

- i. V is monotone of class $*$.
- ii. $V \subset \pi^{-1}(\mathbb{R} \times [\frac{1}{2}, +\infty))$
- iii. V has only negative ends belonging to $\mathcal{L}^*(M)$.

Condition *iii* means in this case that for some point z along one of the rays ℓ_i associated to the ends of V we have that the Lagrangian $V \cap \pi^{-1}(z)$ belongs to $\mathcal{L}^*(M)$. For a fixed ray ℓ_i it is easy to see that this condition does not depend on the choice of the point z .

To define the morphisms and the operations μ_k we proceed as follows. We fix a Lefschetz fibration $\pi : E_\tau \rightarrow \mathbb{C}$ that is tame outside a set U whose interior contains $[x-4, y+4] \times (-1, \infty)$ and coincides with E over $[x-4, y+4] \times [-\frac{1}{2}, \infty)$. Such a fibration exists due to the results from §2.3. Recall from §3.3 the construction of the category $\mathcal{Fuk}^*(E_\tau)$. Each object $V \in \mathcal{Ob}(\mathcal{Fuk}^*(E; \tau))$ corresponds to an object $\bar{V} \in \mathcal{Ob}(\mathcal{Fuk}^*(E_\tau))$ that is obtained, as in Remark 2.3.2, by cutting off the ends of V along the line $\{x-4\} \times \mathbb{R} \subset \mathbb{C}$ and extending them horizontally by parallel transport in the fibration E_τ . It is easy to see that the subcategory of $\mathcal{Fuk}^*(E_\tau)$ that consists of all the objects \bar{V} obtained in this way is quasi-equivalent to $\mathcal{Fuk}^*(E_\tau)$ itself because each object of this larger category is quasi-isomorphic to one of the \bar{V} 's. Notice however that the category $\mathcal{Fuk}^*(E_\tau)$ contains more objects than those of the form \bar{V} , an example is provided in Figure 32. We now put $\text{Mor}_{\mathcal{Fuk}^*(E; \tau)}(V, V') = \text{Mor}_{\mathcal{Fuk}^*(E_\tau)}(\bar{V}, \bar{V}')$ and similarly we define all operations in $\mathcal{Fuk}^*(E; \tau)$ associated to V_1, \dots, V_{k+1} by means of the corresponding operations associated to $\bar{V}_1, \dots, \bar{V}_{k+1}$ in $\mathcal{Fuk}^*(E_\tau)$.

It is clear, by construction, that there is an inclusion:

$$\mathcal{Fuk}^*(E; \tau) \rightarrow \mathcal{Fuk}^*(E_\tau)$$

which is a quasi-equivalence.

The A_∞ -category in the statement of Theorem A can be taken to be any of the categories $\mathcal{Fuk}^*(E; \tau)$ described above. We will see later in the paper that the derived category $D\mathcal{Fuk}^*(E; \tau)$ is independent of τ up to equivalence. Therefore, the omission of τ in the statement of Theorem A is justified.

Remark 3.4.1. We believe that any two A_∞ -categories $\mathcal{Fuk}^*(E; \tau)$ and $\mathcal{Fuk}^*(E; \tau')$ are quasi-equivalent. Indeed, we expect that our construction of the Fukaya category of a tame fibration adapts to the case of a general Lefschetz fibration and the resulting fibration $\mathcal{Fuk}^*(E)$ is

expected to be quasi-equivalent to $\mathcal{Fuk}^*(E; \tau)$ for all τ . The technical ingredients required in the definition of $\mathcal{Fuk}^*(E)$ go beyond the construction in the tame case so that we prefer not to further explore this issue here. In a different direction, we also expect that there is a derived Fukaya category of cobordisms with ends of arbitrary heights in \mathbb{R}^+ and not only with integral heights, as described in this paper. First, given any infinite sequence of strictly increasing positive reals $S = \{a_1, \dots, a_n, \dots\}$ there is a Fukaya category of cobordisms with ends in S that is defined just as in the case of $S = \mathbb{N}^*$. The sets S are ordered by inclusion in an obvious way and this order implies the existence of comparison maps among the corresponding categories. The category in question is expected to be defined as an appropriate limit over S . Again, we do not pursue this construction here as it is not significant for the purpose of this paper.

4. DECOMPOSING COBORDISMS

Fix a Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$ and a Fukaya category $\mathcal{Fuk}^*(E; \tau)$ as defined in §3.4. This section contains the main result of the paper. It claims that each object V of $D\mathcal{Fuk}^*(E; \tau)$ admits an iterated cone decomposition in terms of simpler objects. We will also see later in the paper that $D\mathcal{Fuk}^*(E; \tau)$ is independent of τ .

4.1. Statement of the main result. We will restate here Theorem A after providing the precise definitions of the objects involved.

To fix ideas, we assume that π has m critical points $x_k \in E$, $k = 1, \dots, m$ of corresponding critical values $v_k = (k, \frac{3}{2}) \in \mathbb{C}$. Consider a Fukaya category $\mathcal{Fuk}^*(E; \tau)$ of uniformly monotone negative ended cobordisms $V \subset E$ that are cylindrical outside $\pi^{-1}(S_{x-3, y+3})$ with $x < 0 < y$ and so that all the singularities of π are contained in $\pi^{-1}(S_{x, y})$. See §3.4 for the definition. In particular, τ indicates that the morphisms and operations in $\mathcal{Fuk}^*(E; \tau)$ are defined by means of the Fukaya A_∞ -category $\mathcal{Fuk}^*(E_\tau)$ associated to a tame Lefschetz fibration $\pi : E_\tau \rightarrow \mathbb{C}$ that agrees with E over $[x - 4, y + 4] \times [-\frac{1}{2}, \infty)$.

The objects of $\mathcal{Fuk}^*(E; \tau)$ are collected in the set $\mathcal{L}^*(E)$.

4.1.1. The “atoms” of the decomposition. Our first task is to describe the simpler objects that form the basic pieces of our decomposition.

We will make use of two types of smooth curves in the plane.

- (I) These curves are denoted by γ_i , $i \geq 2$ and are so that $\gamma_i : \mathbb{R} \rightarrow \mathbb{C}$ is a smooth embedding with

$$\gamma_i(\mathbb{R}) \subset (-\infty, x) \times [\frac{1}{2}, +\infty) , \quad \gamma_i(-1, 1) \subset [x - 2, x - 1] \times [1, i]$$

and:

$$\gamma_i((-\infty, -1]) = (-\infty, x - 2] \times \{1\} , \quad \gamma_i([+1, +\infty)) = (-\infty, x - 2] \times \{i\} , \quad .$$

(II) The second type of curve is denoted by t_k . For $1 \leq k \leq m$ the curve t_k is given by a smooth embedding $t_k : (-\infty, 0] \rightarrow \mathbb{C}$ so that we have

$$t_k(0) = v_k, \quad t_k((-\infty, -2]) = (-\infty, x-2] \times \{1\}, \quad t_k((-\infty, 0)) \subset (-\infty, m+1) \times [1, 3]$$

and t_k turns once around all the points $v_{k+1}, v_{k+2}, \dots, v_m$.

Both types of curves are pictured in Figure 10.

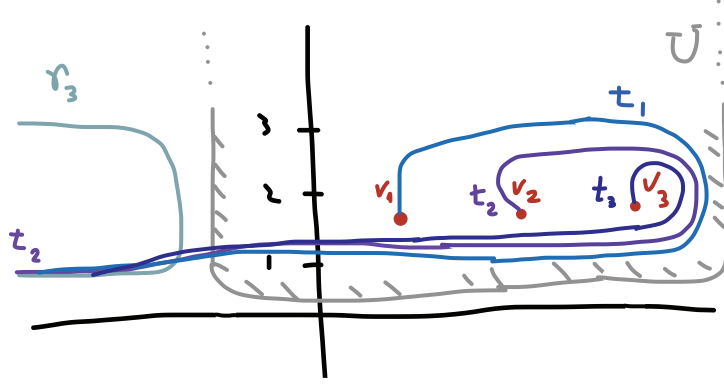


FIGURE 10. The special curves γ_3 and t_1, t_2, t_3 for a fibration E with three critical points.

Let $x-3 < a < x-2$ and fix the points $z_i = (a, i) \in \mathbb{R}^2 \approx \mathbb{C}$, $i \in \mathbb{N}$. Set also $z_* = (a, 1) \in \mathbb{R}^2$ (of course, $z_1 = z_*$, we use this double notation because we want to view z_* as a base-point). Let (M_{z_i}, ω_{z_i}) be the fiber of π over the point z_i . There are two families of Lagrangian cobordisms in $\mathcal{L}^*(E)$ that are associated to the geometric data given above.

- (I') For each Lagrangian in $L \in \mathcal{L}^*(M_{z_i})$ we consider the trail $\gamma_k L$ of L along the curve γ_k . This is a well-defined Lagrangian in E and, further, $\gamma_k L \in \mathcal{L}^*(E)$.
- (II') Denote by T_i the thimble associated to the singularity x_i and the curve t_i . Denote by $S_i \subset M_{z_*}$ the vanishing sphere associated to the singularity x_i such that T_i is the trail of S_i along t_i . Since E is strongly monotone it follows from Proposition 3.2.3 that $T_i \in \mathcal{L}^*(E)$.

4.1.2. *The decomposition.* We now reformulate Theorem A in the setting and notation above. Recall that we use the Novikov ring \mathcal{A} as coefficients at all times.

Theorem 4.1.1 (Theorem A reformulated). *Let $V \in \mathcal{L}^*(E)$ be a Lagrangian with s cylindrical ends $L_i = V|_{z_i}$, $1 \leq i \leq s$ (as in Definition 2.2.1). There exist finite rank \mathcal{A} -modules E_k , $1 \leq k \leq m$, and an iterated cone decomposition taking place in $D\mathcal{Fuk}^*(E; \tau)$:*

$$V \cong (T_1 \otimes E_1 \rightarrow T_2 \otimes E_2 \rightarrow \dots \rightarrow T_m \otimes E_m \rightarrow \gamma_s L_s \rightarrow \gamma_{s-1} L_{s-1} \rightarrow \dots \rightarrow \gamma_2 L_2) .$$

Moreover, the category $D\mathcal{Fuk}^*(E; \tau)$ is independent of τ (up to equivalence).

The proof of Theorem 4.1.1 follows from an analogue result - Theorem 4.2.1, stated in the first subsection below - which applies to tame Lefschetz fibrations. The three subsequent subsections §4.3 - §4.5 form the technical heart of the paper. They provide the arguments that are put together in §4.6 to show Theorem 4.2.1. The decomposition in the statement of Theorem 4.1.1 follows directly from that provided by Theorem 4.2.1. The modules E_i are explicitly identified along the proof - see equation (57). The independence of $D\mathcal{Fuk}^*(E; \tau)$ from the choice of τ is postponed to §5 as it is an immediate consequence of Corollary 5.1.3 which is itself deduced from Theorem 4.2.1.

4.2. Decomposition of cobordisms in tame fibrations. Assume now that the Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$ is tame outside the set U - as in Definition 2.2.2 - and is so that:

- i. the set U contains $[0, m+1] \times [\frac{1}{2}, K]$ and, as in (18), $U \subset \mathbb{R} \times [0, +\infty)$.
- ii. as before, π has m critical points $x_k \in E$ of corresponding critical values $v_k = (k, \frac{3}{2})$.
- iii. we fix $a_U > 0$ sufficiently large so that the set $\{z \pm d \mid z \in U, d \in [0, 4] \subset \mathbb{R}\}$ is disjoint from both quadrants

$$Q_U^- = (-\infty, -a_U] \times [0, +\infty), \quad Q_U^+ = [a_U, \infty) \times [0, +\infty).$$

In this setting we again first define the “simple” pieces that appear in the relevant decomposition. They again involve two types of curves, again denoted by γ_i and t_j , and are defined as at the points (I) and (II) in §4.1.1 but by using instead of the constant x the value $-a_U + 3$. As a consequence, the position of these curves relative to the set U is as in Figure 10. With this definition we then define the two families of associated Lagrangians as at the points (I') and (II'). Notice that the Lagrangian $\gamma_k L$ is a product $\gamma_k L = \gamma_k \times L$. This is because the fibration is trivial over the complement of U and γ_k is entirely contained in this complement. At the same time, because of condition iii above, $\gamma_k L$ as well as T_j are cobordisms in the sense of Definition 2.2.3 (relative to the constant a_U). Finally, assume that $L \in \mathcal{L}^*(M)$. Thus the $\gamma_k L$'s are objects of $\mathcal{L}^*(E)$, and by Proposition 3.2.3 the same holds for the T_j 's.

We reformulate again Theorem A in this context:

Theorem 4.2.1. *Let $V \in \mathcal{L}^*(E)$, $V : \emptyset \rightarrow (L_1, \dots, L_s)$. There exist finite rank \mathcal{A} -modules E_k , $1 \leq k \leq m$, and an iterated cone decomposition taking place in $D\mathcal{Fuk}^*(E)$:*

$$V \cong (T_1 \otimes E_1 \rightarrow T_2 \otimes E_2 \rightarrow \dots \rightarrow T_m \otimes E_m \rightarrow \gamma_s \times L_s \rightarrow \gamma_{s-1} \times L_{s-1} \rightarrow \dots \rightarrow \gamma_2 \times L_2) .$$

4.3. Decomposition of remote Yoneda modules. In this subsection we assume the “tame” setting of §4.2 and we consider a particular class of A_∞ -modules over $\mathcal{Fuk}^*(E)$ associated to certain cobordisms W included in Lefschetz fibrations that extend E .

Specifically, fix a large constant $K > 0$ and consider a Lefschetz fibration $\hat{\pi} : \hat{E} \rightarrow \mathbb{C}$ so that:

- i. $\hat{\pi}$ is tame outside \hat{U} , with $U \subset \hat{U}$ and is so that condition (4) is satisfied for some constant $a_{\hat{U}} > a_U$.
- ii. $\hat{U} \subset \mathbb{R} \times [-K, +\infty)$.
- iii. $\hat{E}|_{\mathbb{R} \times [-\frac{1}{2}, +\infty)} = E|_{\mathbb{R} \times [-\frac{1}{2}, +\infty)}$ including their symplectic structures.

Similarly to the definition of the category $\mathcal{Fuk}^*(E)$ in §3.3 we consider a Fukaya category $\mathcal{Fuk}^*(\hat{E})$ whose objects are cobordisms $W \subset \hat{E}$ as in Definition 2.2.3 so that W is monotone of class $* = (\rho, d)$, W has only negative ends L_1, \dots, L_s (all in $\mathcal{L}^*(M)$) and, similarly to ii in §3.3,

$$W \subset \hat{\pi}^{-1}(\mathbb{R} \times [-K + \frac{1}{2}, \infty)) .$$

Following Definition 2.2.3, the cobordism W is cylindrical and the ends of W project to rays of the form $(-\infty, -a_U] \times \{k\}$ with $k \in \mathbb{N}^*$.

A cobordism W as before is called *remote* relative to E if, in addition,

$$(25) \quad W \subset \hat{\pi}^{-1}(\mathbb{R} \times (-\infty, 0] \cup Q_U^-) .$$

In this case, we deduce, in particular, that $W \cap \pi^{-1}(U) = \emptyset$ (this explains the terminology, in the sense that W is remote from all the singularities of π). See Figure 11. It is important to note that because \hat{U} might contain an unbounded region disjoint from the upper half plane (in the figure this region goes through the third quadrant, it could as well also intersect the fourth quadrant but that is irrelevant for the argument), the conditions i,ii,iii allow for \hat{E} to have more singularities than E .

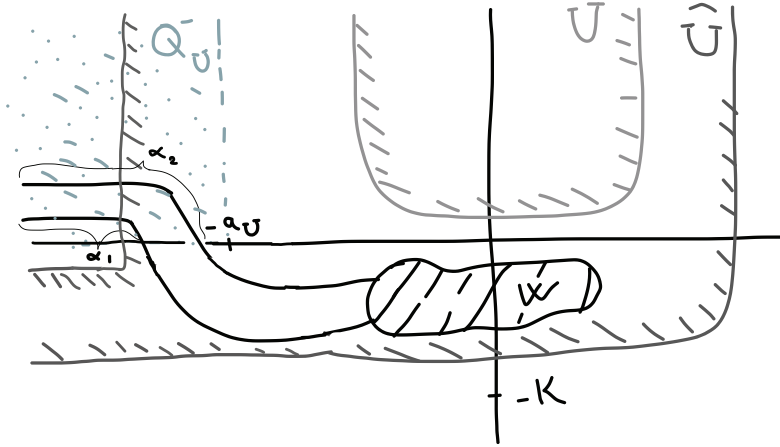


FIGURE 11. The domains \hat{U} , U , the quadrant Q_U^- and the cobordism W that is remote relative to E .

Given property ii from §3.3, it is clear that such remote cobordisms W are not objects of $\mathcal{Fuk}^*(E)$. On the other hand, each object of $\mathcal{Fuk}^*(E)$ is an object of $\mathcal{Fuk}^*(\hat{E})$. Moreover, by a simple application of the open mapping theorem, we see that there is an inclusion of A_∞ -categories

$$(26) \quad \text{Incl}^{E, \hat{E}} : \mathcal{Fuk}^*(E) \rightarrow \mathcal{Fuk}^*(\hat{E}) .$$

The relevant argument is as follows. All objects of $\mathcal{Fuk}^*(E)$ project to the upper half plane so that the J -polygons that compute the operations μ^k of $\mathcal{Fuk}^*(\hat{E})$ (for objects that are in $\mathcal{Fuk}^*(E)$) project to curves v in \mathbb{C} with boundary inside the upper half plane. Our choice of almost complex structures imply that such a curve v can be assumed - after applying the change of coordinates as in §3.3.4 - to be holomorphic outside (possibly a slightly bigger set containing) U and, by the open mapping theorem, we deduce that v can not extend outside of the region where E and \hat{E} coincide. Thus, for objects picked in $\mathcal{Fuk}^*(E)$, the operations μ_k are the same in $\mathcal{Fuk}^*(\hat{E})$ and in $\mathcal{Fuk}^*(E)$.

Let $\mathcal{Y}(W)$ be the Yoneda module associated to an object $W \in \mathcal{Ob}(\mathcal{Fuk}^*(\hat{E}))$. We denote by W_E the pull-back module:

$$(27) \quad W_E = (\text{Incl}^{E, \hat{E}})^*(\mathcal{Y}(W))$$

In case W is remote with respect to E we say that the module W_E is a remote $\mathcal{Fuk}^*(E)$ -module.

Proposition 4.3.1. *With the terminology above, assume that $W \in \mathcal{Ob}(\mathcal{Fuk}^*(\hat{E}))$ is remote relative to E , $W : \emptyset \rightsquigarrow (L_1, \dots, L_s)$, then $W_E \in \mathcal{Ob}(D\mathcal{Fuk}^*(E))$ and it admits a decomposition in $D\mathcal{Fuk}^*(E)$ of the following form:*

$$(28) \quad W_E = (\gamma_s \times L_s \rightarrow \gamma_{s-1} \times L_{s-1} \rightarrow \dots \rightarrow \gamma_2 \times L_2)$$

To unwrap a bit the meaning of this Proposition consider a cobordism W in E . If there is a horizontal hamiltonian isotopy $\phi : \hat{E} \rightarrow \hat{E}$ that pushes W away from the singularities of π , in the sense that $\pi(\phi(W)) \cap U = \emptyset$, then the Proposition implies that W admits a decomposition as claimed in Theorem 4.2.1 but with all the modules $E_i = 0$. As a particular case that is already of interest, if π has no singularities $E = \mathbb{C} \times M$ ($U = \emptyset$ and $m = 0$), then Proposition 4.3.1 applies to any cobordism $W \subset E = \mathbb{C} \times M$. Thus, for $E = \mathbb{C} \times M$, Proposition 4.3.1 implies Theorem 4.2.1.

Remark 4.3.2. In this paper we mostly assume that our Lefschetz fibrations are strongly monotone, which in turn determines a monotonicity class $*$ for the associated Fukaya categories. However, Proposition 4.3.1 continues to hold for remote cobordisms of arbitrary monotonicity classes $*$ (subject to the restrictions on $*$ made on page 21 in §3.1). The point is that we can analyze remote cobordisms as if they live in a trivial Lefschetz fibration, and so there is no

need to take into account monotonicity properties of the thimbles and vanishing spheres. See the “exception” to Definition 3.2.1 on page 25.

Proof of Proposition 4.3.1. We start by repositioning W by using a horizontal Hamiltonian isotopy in \hat{E} . By definition, this is an isotopy possibly not with compact support, whose support contains a neighborhood of the singularities of \hat{E} , and which slides the ends of W along themselves just as in Definition 2.2.3 in [BC3]. It is immediate to see that such isotopies do not change the isomorphism type of objects in $\mathcal{Fuk}^*(\hat{E})$.

By applying such an isotopy to W we may assume that not only $W \subset \hat{\pi}^{-1}(\mathbb{R} \times (-\infty, 0] \cup Q_U^-)$ as in the definition of remote cobordisms but that, moreover, the intersection

$$W^- = W \cap Q_U^-$$

coincides with a disjoint union of cylindrical ends of W . In other terms

$$W^- = \cup_{i=1}^s \alpha_i \times L_i$$

where α_i are curves in \mathbb{C} as in Figure 11. In particular, for any object $X \in \mathcal{Ob}(\mathcal{Fuk}^*(E))$, the intersection $W \cap X$ consists of a union of intersections of the ends of W with the ends of X and is included in the quadrant Q_U^- .

The main part of the proof makes essential use of constructions that appear in [BC3]. It consists of three main steps.

Step 1: Repositioning W . Here we replace the module W_E with a quasi-isomorphic module corresponding to a cylindrical Lagrangian that can be handled easier geometrically. For this purpose we include the two A_∞ -categories $\mathcal{Fuk}^*(E)$ and $\mathcal{Fuk}^*(\hat{E})$ in two other A_∞ -categories, respectively, $\mathcal{Fuk}_{\frac{1}{2}}^*(E)$ and $\mathcal{Fuk}_{\frac{1}{2}}^*(\hat{E})$. These two categories have objects that are again cobordisms as before with the difference that their ends have heights $\in \frac{1}{2}\mathbb{Z} \subset \mathbb{Q}$. In other words, compared with Definition 2.2.3, the difference is that $V \cap \pi^{-1}(Q_U^-) = \cup_{i \in \mathbb{N}^*} ((-\infty, -a_U] \times \{\frac{i}{2}\}) \times L_i$. The inclusion $\mathcal{Fuk}^*(E) \rightarrow \mathcal{Fuk}_{\frac{1}{2}}^*(E)$ is obvious and is clearly full and faithful and similarly for the two categories associated to \hat{E} . We now perturb W by a (non-horizontal) Hamiltonian isotopy so as to obtain an object W' of $\mathcal{Fuk}_{\frac{1}{2}}^*(\hat{E})$ that differs from W only inside $(-\infty, -a_U - 2] \times [\frac{1}{2}, +\infty)$ and is so that the ends of W' restricted to $(-\infty, -a_U - 4 - s] \times [\frac{1}{2}, +\infty)$ are of the form $(-\infty, -a_U - 4 - s] \times \{i - \frac{1}{2}\} \times L_i$ (for all the definitions involved to be coherent we might need to enlarge here the set \hat{U}). In other words, the ends of W' are shifted down by $\frac{1}{2}$ compared to the ends of W . Let W'_E be the $\mathcal{Fuk}^*(E)$ -module obtained as pull-back over the inclusions

$$\mathcal{Fuk}^*(E) \rightarrow \mathcal{Fuk}^*(\hat{E}) \rightarrow \mathcal{Fuk}_{\frac{1}{2}}^*(\hat{E})$$

from the $\mathcal{Fuk}_{\frac{1}{2}}^*(\hat{E})$ -module $\mathcal{Y}(W')$. The two modules W_E and W'_E are quasi-isomorphic.

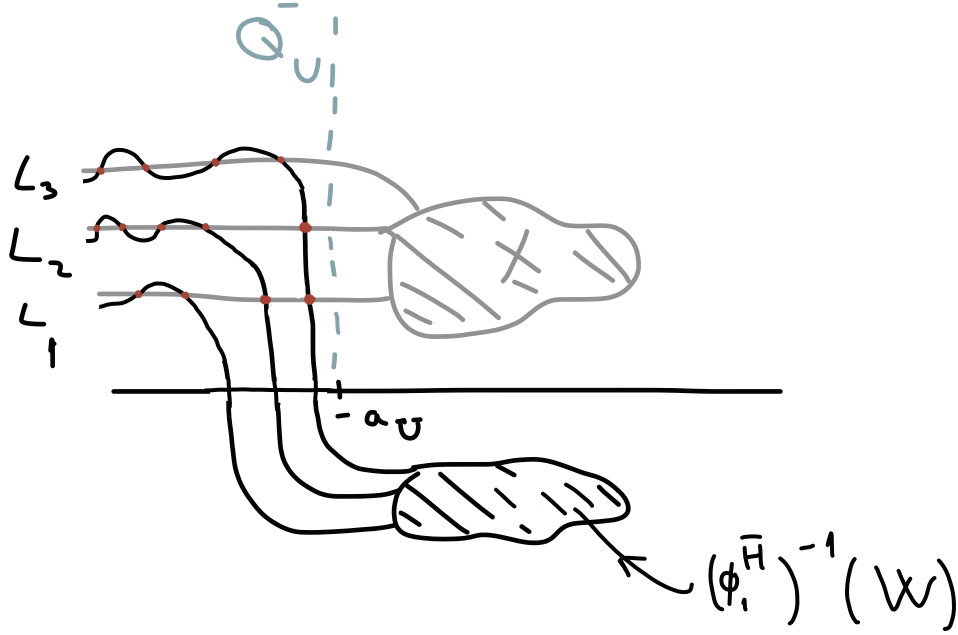


FIGURE 12. The projections on \mathbb{C} of $(\phi_1^{\bar{H}_{X,W}})^{-1}(W)$ and of X . The ends of $(\phi_1^{\bar{H}_{X,W}})^{-1}(W)$ are below those of X at infinity.

This is a direct consequence of the definition of $\text{Mor}_{\mathcal{F}uk^*(\hat{E})}(X, W) = CF(X, W)$. This uses a perturbation of W in which its negative ends are “moved” down compared to those of X . More precisely, recall from §3 in [BC3] (see also Figure 8 there) that $CF(X, W)$ is defined by using a specific profile function h and an associated Hamiltonian $\bar{H}_{X,W}$. With these choices $CF(X, W)$ is identified with $CF(X, (\phi_1^{\bar{H}_{X,W}})^{-1}(W))$ (under the assumption that X and $(\phi_1^{\bar{H}_{X,W}})^{-1}(W)$ intersect transversely). The projection of $(\phi_1^{\bar{H}_{X,W}})^{-1}(W)$ to \mathbb{C} is as in Figure 12. On the other hand the ends of W' are, by construction, below the horizontal lines $\mathbb{R} \times \{i\}$ and therefore the complexes $CF(X, W)$ and $CF(X, W')$ are quasi-isomorphic. Further, this quasi-isomorphism extends to a quasi-isomorphism of the modules W_E and W'_E .

To summarize this first step, we have replaced in our argument the cobordism W by the cobordism W' . Moreover, by a further horizontal Hamiltonian isotopy, we may assume that W' has a projection as in Figure 13. More precisely, we assume that $(W')^- = W' \cap Q_U^-$ is a disjoint union of components $\alpha_i \times L_i$ so that α_i is obtained by rounding the corner of the union of two intervals $(-\infty, -a_U - 4 - s + i] \times \{i - \frac{1}{2}\} \cup \{-a_U - 4 - s + i\} \times [0, i - \frac{1}{2}]$. In particular, the intersections of X and W' project onto \mathbb{C} to the points $b_{ij} = \{-a_U - 4 - s + i\} \times \{j\}$ with $i > j$, $i, j \in \mathbb{N}^*$, $i = 1, 2, \dots, s$; b_{ij} is precisely the projection of the intersection of the i -th end of W' with the j -th end of X .

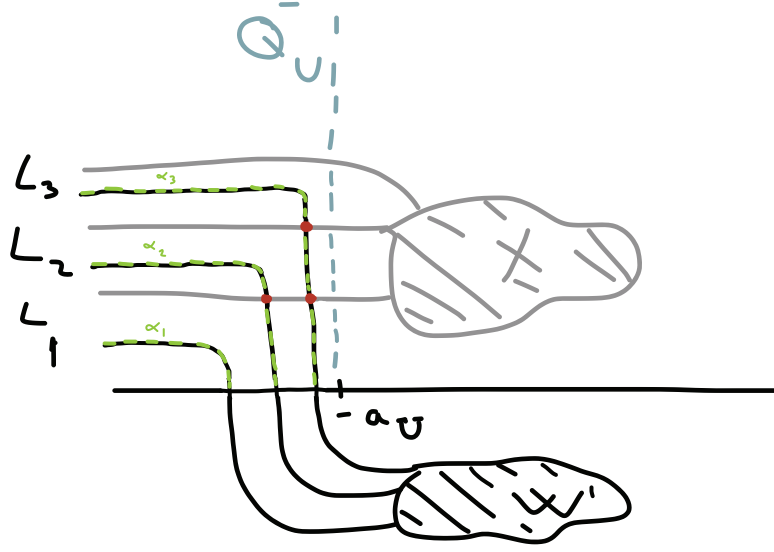


FIGURE 13. The remote cobordism $W' \subset \hat{E}$, the object $X \in \mathcal{Ob}(\mathcal{Fuk}^*(E))$ and the curves α_i . The height of the i -th end of W' is $i - \frac{1}{2}$ while the i -th end of X has height i .

We may also assume, by a slight additional horizontal isotopy, that $W' \cap \pi^{-1}(\mathbb{R} \times [-\frac{1}{2}, \infty))$ is a union of cylindrical ends.

Step 2 : “Snaky” perturbation data. This step of the proof consists in choosing the perturbation data used in the definition of $\mathcal{Fuk}^*(E)$ and $\mathcal{Fuk}^*(\hat{E})$ in a convenient way. Recall that W' is already fixed as discussed at step 1. The perturbation data in question are chosen as described in §3.3 except that the profile function h as well as the almost complex structure \mathbf{J} will be picked with some additional properties described below.

We start with the choice of the profile function h . As can be seen from §3.3 the fundamental ingredients in the definition of h are the functions h_{\pm} . We start with h_{+} : the only requirement in this case is that $h_{+} : [a_U + \frac{3}{2}, \infty) \rightarrow \mathbb{R}$ has its single critical point (the bottleneck) at $a_U + 2$. In other words the difference with respect to the construction at §3.3.2 is that the value $\frac{1}{2}$ is replaced with a_U . In fact, as we only consider cobordisms without positive ends the choice of h_{+} is not particularly important as long as the bottlenecks are away from U . We now discuss the function h_{-} . This is a smooth function $h_{-} : (-\infty, -a_U - 1] \rightarrow \mathbb{R}$ with the following additional properties - see Figure 14:

- a'. The function h_{-} has critical points $o_i = -a_U - 3 - i$, $i = 0, 1, \dots, s$ that are non-degenerate local maxima.

- a''. The function h_- has critical points $o'_i = -a_U - \frac{7}{2} - i$, $i = 0, 1, \dots, s-1$ that are non-degenerate local minima.
- a'''. h_- has no other critical points than those at a', a'' above and for all $x \in (-\infty, a_U - 4 - s]$ we have $h_-(x) = \alpha^- x + \beta^-$ for some constants $\alpha^-, \beta^-, \alpha^- > 0$.

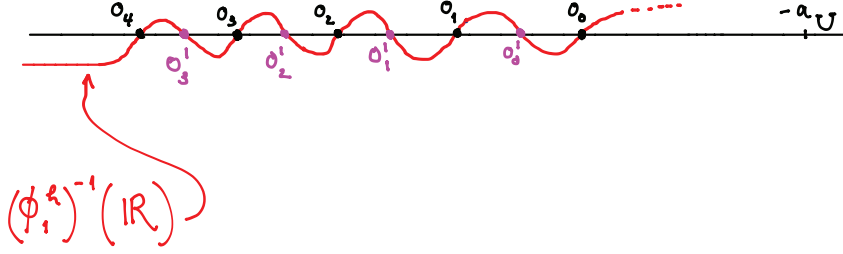


FIGURE 14. The graph of $(\phi_1^h)^{-1}(\mathbb{R})$ for $s = 4$.

Beyond this, the properties of the function h are obtained by direct analogy with those given at the points i, ii, iii, iv in §3.3.2 but with the point a replaced by the three conditions a', a'', a''' above. In particular, the set W_i^- now becomes $W_i^- = (-\infty, -a_U - 1] \times [i - \epsilon, i + \epsilon]$ and $T_i^- = (-\infty, -a_U - 1] \times [i - \epsilon/2, i + \epsilon/2]$. From this point on, the construction continues along the same approach as in §3.3. In particular, the properties of the family Θ and those of \mathbf{J} are just the same as properties a, b, c in §3.3.3 but they are relative to sets $K_{V_1, \dots, V_{k+1}}$ that satisfy different requirements compared to those in §3.3.3.

We now discuss the two properties required of $K_{V_1, \dots, V_{k+1}}$. We start by underlining that, because we care here about a module structure, while V_1, \dots, V_k are elements of $\mathcal{L}^*(E)$, V_{k+1} is either an element of $\mathcal{L}^*(E)$ or $V_{k+1} = W'$. Further, we fix small disks $D_{ij} \subset \mathbb{C}$ of radius smaller than $\frac{1}{8}$ that are respectively centered at the points (o'_i, j) , $i = 0, \dots, s-1$, $j \in \{1, \dots, s_{V_1, \dots, V_{k+1}}\}$. We denote by $D'_{ij} \subset D_{ij}$ the disk with the same center but with radius half of that of D_{ij} . Recall, that $s_{V_1, \dots, V_{k+1}}$ is the smallest $l \in \mathbb{N}$ so that $\pi(V_1 \cup V_2 \cup \dots \cup V_{k+1}) \subset [\frac{1}{2}, l)$. We also pick a compact set $Z \subset \mathbb{R} \times (-\infty, -\frac{1}{4}]$ which contains in its interior $\pi(W') \cap \mathbb{R} \times (-\infty, -\frac{1}{2}]$ (recall that W' is cylindrical outside $\pi^{-1}(\mathbb{R} \times (-\infty, -\frac{1}{2}])$) as well as a slightly bigger set $Z' \subset \mathbb{R} \times (-\infty, -\frac{1}{4}]$. We require:

$$(29) \quad K_{V_1, \dots, V_{k+1}} \supset \cup_{i,j} D'_{ij} \cup [-a_U - \frac{11}{4}, a_U + \frac{7}{4}] \times [\frac{1}{4}, s_{V_1, \dots, V_{k+1}} + 1] \cup Z.$$

and

$$(30) \quad K_{V_1, \dots, V_{k+1}} \subset \cup_{i,j} D_{ij} \cup [-a_U - \frac{13}{4}, a_U + 2] \times [\frac{1}{8}, +\infty) \cup Z'.$$

We now will see that this class of perturbation data is sufficient to insure the regularity and the compactness of the moduli spaces appearing in the definition of the category $\mathcal{Fuk}^*(E)$ and of the $\mathcal{Fuk}^*(E)$ -module W'_E . In the next section we will use these specific perturbations to extract the exact triangles claimed in the statement.

Let $u : S_r \rightarrow E$ be a solution of (20) that satisfies the boundary and asymptotic conditions required to define the multiplications μ_k for $\mathcal{Fuk}^*(E)$ or for the definition of the module W_E . In the first case the boundary conditions are along cobordisms V_1, \dots, V_{k+1} ($V_i \in \mathcal{L}^*(E)$, in particular, V_i projects on the upper half plane). In the second case, the curve is defined on a punctured polygon so that the component C_i of the boundary of the polygon is mapped to V_i for $1 \leq i \leq k$ and the $k+1$ -th component C_{k+1} is mapped to W' .

By the change of variables in §3.3.4, (and by taking h sufficiently small) we deduce that there exists some small $\delta > 0$ so that if $u : S_r \rightarrow E$ satisfies (20) with the choice of perturbation data as just above and if $v : S_r \rightarrow E$ is defined by $u(z) = \phi_{a_r(z)}^{\bar{h}}(v(z))$, then $v' = \pi \circ v$ is holomorphic outside of the set

$$(31) \quad \widehat{K} = \cup_{i,j} D''_{ij} \cup [-a_U - \frac{13}{4} - \delta, a_U + 2 + \delta] \times [\frac{1}{8} - \delta, +\infty) \cup Z'',$$

where D''_{ij} is a disk with the same center as D_{ij} but slightly bigger and, similarly, Z'' is a set slightly bigger than Z' - see Figure 15. In view of this transformation, compactness for the relevant moduli spaces follows without difficulty by the usual bottleneck argument §3.3 [BC3]. Thus, the only issue that requires some attention is regularity. Denote

$$K' = \cup_{i,j} D'_{ij} \cup [-a_U - \frac{11}{4}, a_U + \frac{7}{4}] \times [\frac{1}{4}, s_{V_1, \dots, V_{k+1}} + 1] \cup Z.$$

Given that $K' \subset K_{V_1, \dots, V_k}$, the perturbation data can be chosen freely over K' and thus, for all moduli spaces consisting of curves whose image intersects $\pi^{-1}(K')$ regularity can be handled in the standard fashion as in [Sei3]. Therefore, we are left to analyze the curves $u : S_r \rightarrow E$ so that $\pi(u)$ has an image disjoint from K' . Assume first that u appears in the definition of the higher structures of $\mathcal{Fuk}^*(E)$. In this case, the condition $\pi^{-1}(K') \cap \text{Image}(u) = \emptyset$ implies that all the boundary of u projects onto \mathbb{C} along a single line $(-\infty, -a_U - 2] \times \{j\}$. Given that $(o'_i, j) \in K'$, it follows that the image of $\pi(u)$ can not cross any of the points (o'_i, j) , nor can it have one of these points as asymptotic limit. As a consequence, the asymptotic limits of u have to project to just one of the points (o_i, j) . But by now taking a look to v' which is holomorphic around (o_i, j) one sees immediately that v' and thus $\pi(u)$ has to be constant (indeed, (o_i, j) can not be the exit point of v' by an application of the open mapping theorem). The second possibility to consider is if u appears in the definition of the module structure of W'_E . It is immediate, in this case too that $\pi^{-1}(K') \cap \text{Image}(u) = \emptyset$ implies that all asymptotic limits of u coincide with a single point b_{ij} (which is, of course, also of the form (o_i, j)). It is

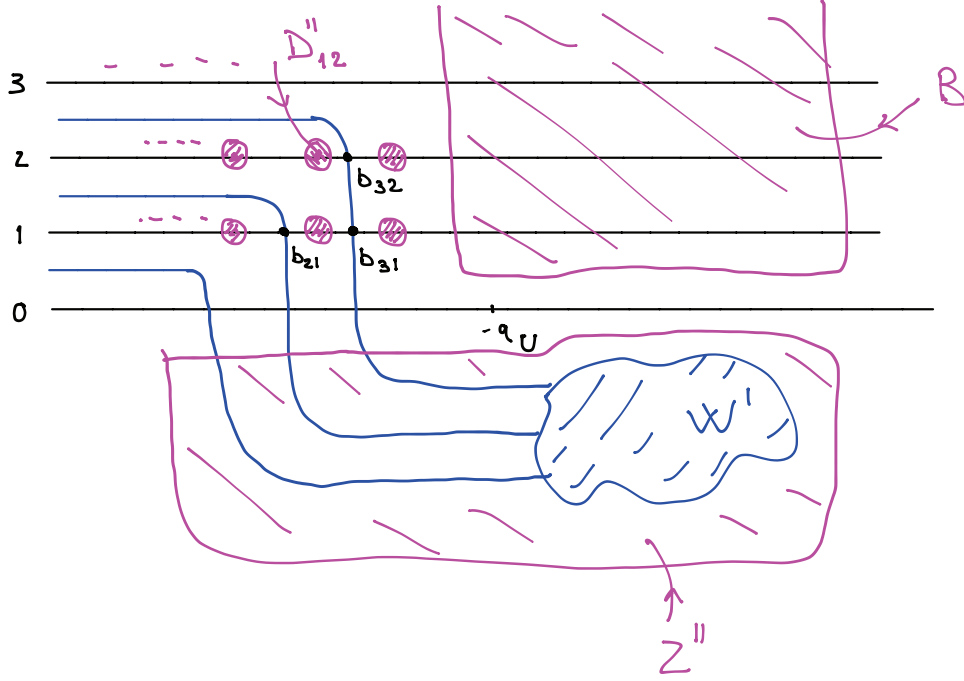


FIGURE 15. The set \widehat{K} outside which v' is holomorphic is the union of all the regions in pink: the disks D''_{ij} , the box

$$B = [-a_U - \frac{13}{4} - \delta, a_U + 2 + \delta] \times [\frac{1}{8} - \delta, +\infty)$$

and the neighborhood Z'' of the non-cylindrical part of $\pi(W')$. Are also pictured the points b_{ij} . Here $s = 3$. The non cylindrical part of the cobordisms $X \in \mathcal{L}^*(E)$ projects inside B .

easy to see by an application of the open mapping theorem that in this case $\pi(u)$ has again to be constant. To conclude this argument, the only moduli spaces for which regularity is in question consist of curves u so that $\pi(u)$ is constant equal to one of the point (o_i, j) . That means that these curves take values in the fiber over (o_i, j) and, because o_i is a local maximum of h_- , one can see, as in §4.2 [BC3] that by picking regular data in the fiber these moduli spaces are regular too.

Thus the regularity of all the moduli spaces involved can be achieved by generic choices of data. We work from now on with such data associated to the “snaky” perturbations constructed at this step.

Step 3: *The proof of (28).* We will show now that there is a sequence of $\mathcal{Fuk}^*(E)$ -modules \tilde{L}_i , $W'_{E,i}$, $i = 1, \dots, s$, with $W'_{E,i}$ being submodules of W'_E , so that:

- i. $W'_{E,1} = 0$, $W'_{E,s} = W'_E$ and for $i \geq 2$ there exist exact sequences of $\mathcal{Fuk}^*(E)$ -modules

$$0 \rightarrow W'_{E,i-1} \rightarrow W'_{E,i} \rightarrow \tilde{L}_i \rightarrow 0$$

- ii. there exists a quasi-isomorphism of $\mathcal{Fuk}^*(E)$ -modules

$$\tilde{L}_i \simeq \mathcal{Y}(\gamma_i \times L_i),$$

where \mathcal{Y} is the Yoneda embedding for $\mathcal{Fuk}^*(E)$.

These points immediately imply the statement of Proposition 4.3.1. We now proceed to the construction of $W'_{E,i}$ and to prove the points i, ii above.

Let $X \in \mathcal{L}^*(E)$ and let W' be the remote cobordism as discussed at the first step. We now assume “snaky” perturbations picked as described at the second step. In particular, the complex $CF(X, W')$ is well defined. The generators of this complex are identified with the intersection $X \cap (\phi_1^{\bar{h}})^{-1}(W')$. Notice that due to the choice of snaky perturbations $\pi(X \cap (\phi_1^{\bar{h}})^{-1}(W')) = \pi(X \cap (\phi_1^{\bar{h}})^{-1}(W')) = \{b_{rs}\}_{r,s}$ see Figure 16. We now put

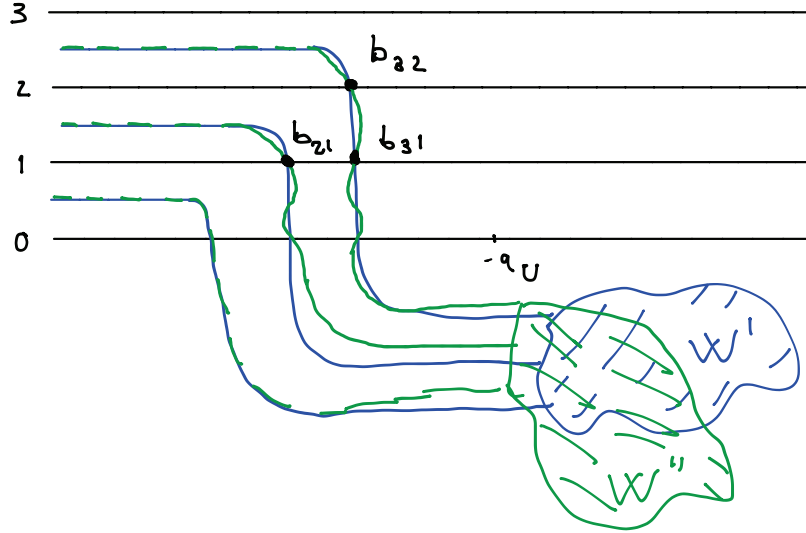


FIGURE 16. The cobordism W' and its perturbation $W'' = (\phi_1^{\bar{h}})^{-1}(W')$.

$$P_{rs}(X) = X \cap (\phi_1^{\bar{h}})^{-1}(W') \cap \pi^{-1}(b_{rs})$$

and we define

$$W'_{E,i}(X) = \mathcal{A}\langle \cup_{1 \leq r \leq i; s < r} P_{rs} \rangle \subset CF(X, W') .$$

In other words, the generators of $W'_{E,i}(X)$ are the intersection points of X with the first i branches of the W' . It is clear from the construction that $W'_{E,1} = 0$ and that $W'_{E,s} = W'_E$. We

will show now that, for each $1 \leq i \leq s$, the structural maps μ_k of W'_E when restricted to $W'_{E,i}$ have values into $W'_{E,i}$. In other words

$$(32) \quad \mu_k|_{W'_{E,i}} : CF(V_1, V_2) \otimes \dots \otimes CF(V_{k-1}, V_k) \otimes W'_{E,i}(V_k) \rightarrow W'_{E,i}(V_1) .$$

This property immediately implies that the $W'_{E,i}$ are indeed A_∞ -modules and moreover that the inclusions of vector spaces $W'_{E,i-1}(-) \subset W'_{E,i}(-)$ are actually inclusions of $\mathcal{Fuk}^*(E)$ -modules. The modules \tilde{L}_i defined as the respective quotients. With these definition for $W'_{E,i}$ and assuming (32), point ii follows because the quotient \tilde{L}_i is naturally identified (up to quasi-isomorphism) with $\mathcal{Y}(\gamma_i \times L_i)$. In summary, to conclude the proof of the proposition it remains to show (32).

Our argument is based on properties of the curve $v' = \pi(v)$ where v is related to a curve $u : S_r \rightarrow E$ by equation (21) and u is a solution of (20) contributing to the module structural map μ_k . Here S_r is the disk with $k+1$ boundary punctures, of which k are the entries and the last one is an exit puncture. The last entry, denoted m , is the “module” entry and is asymptotic to a generator of $CF(V_{k-1}, W'_{E,i})$. The exit, denoted e , is asymptotic to a generator of $CF(V_1, W'_{E,i})$.

We will make the following simplifying assumption: we assume that the transition functions used in the definition of moduli spaces associated to the module operations are so that:

$$(33) \quad a_r(z) = 1 \quad \forall z \in C_{k+1},$$

where C_{k+1} is the component of the boundary of the punctured disk S_r that joins m to e . (See Figure 7 for an illustration of the case $k=3$, where C_4 bounds both ϵ_3 and ϵ_4 .) In other words we use transition functions as in §3.3.1 except that we add (33) and we modify conditions *i. c* and *ii. c'* in §3.3.1 such as to no longer require $a_r \circ \epsilon(s, t) = 0$ for $(s, t) \in \{0\} \times [0, 1]$ for ϵ for the strip like ends associated to m and to e . By imposing (33) *just to the moduli spaces appearing in the definition of modules* over $\mathcal{Fuk}^*(E)$ (and not to those defining the multiplication in $\mathcal{Fuk}^*(E)$ itself) we easily see that, on one hand, condition (33) is compatible with gluing and splitting and, moreover, it does not contradict the definition of the operations in $\mathcal{Fuk}^*(E)$ itself. At the same time, this means that we get two presumptive definitions for the Yoneda modules of objects in $\mathcal{Fuk}^*(E)$: one using the conditions in §3.3.1 and the other making use of (33). However, it is easy to see that the two resulting modules are quasi-isomorphic and thus our simplifying condition does not affect any further arguments.

The geometric advantage of this simplifying assumption on a_r is that v no longer satisfies a moving boundary condition along C_{k+1} , rather v maps all of C_{k+1} to $W'' = (\phi_1^h)^{-1}(W')$. We also remark that, by the definition of h , and the position of $\pi(W')$ relative to the ends of

cobordisms $\in \mathcal{L}^*(E)$ - as in Figure 16 - we have that W'' is just a close perturbation of W' and $\pi(W'')$ intersects the horizontal lines of positive, integral imaginary coordinates transversely and in the same points as $\pi(W')$.

Our claim (32) reduces to showing that if $v'(m) = b_{\alpha\beta}$ and $v'(e) = b_{rs}$, then $r \leq \alpha$.

We first fix some notation relative to certain regions in Q_U^- . First we denote by F the region given as

$$F = \bigcup_{0 \leq t \leq 1, j \in \mathbb{Z}} \phi_{-t}^h((-\infty, -a_U] \times \{j\}) \cup W'' .$$

In short, F is the set swiped by all the potential boundary conditions of the curves v' . Further, we denote $\hat{F} = F \cup \hat{K}$ (see (31)) and we put $G = \mathbb{C} \setminus \hat{F}$ - see Figure 17.

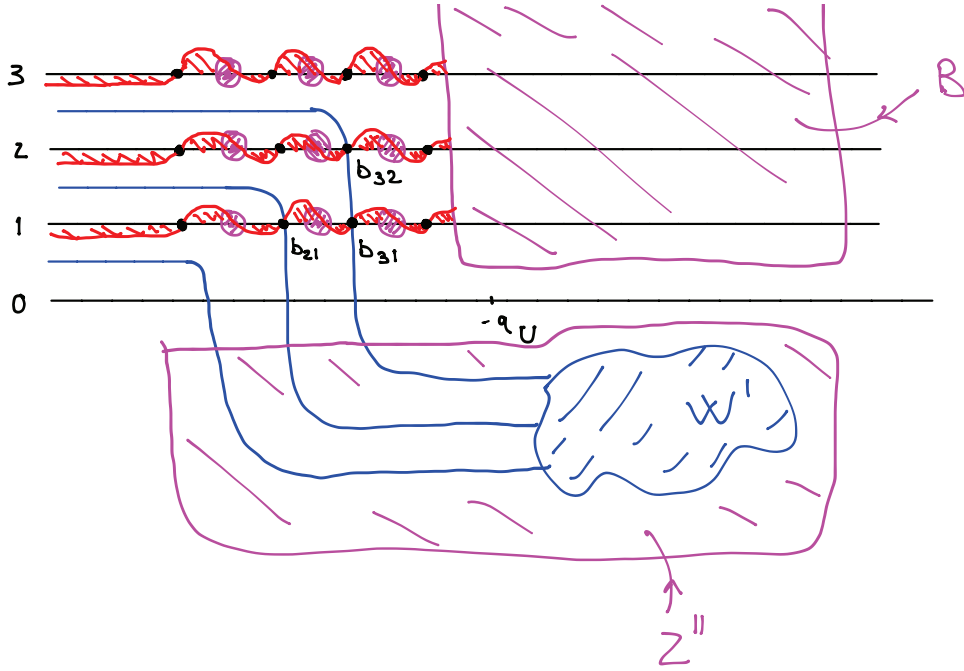


FIGURE 17. The region \hat{F} is the union of \hat{K} (the union of all the pink regions) and F (the region in red).

From step 2 we know that v' is holomorphic over G and clearly, the boundary of S_r is so that $v'(\partial S_r) \cap G = \emptyset$. It is an elementary fact (see for instance Proposition 3.3.1 in [BC3]) that as soon as $Image(v')$ intersects a connected component of G , the full component has to be contained in $Image(v')$. In particular, this means that $Image(v')$ can not intersect an unbounded component of G .

Each point b_{ij} is in the closure of four components of G that meet, basically, as four quadrants at b_{ij} . Our argument will make use of the following:

Lemma 4.3.3. *Suppose that b_{ij} is different from both $v'(e)$ and $v'(m)$ and that the component corresponding to the fourth quadrant at b_{ij} is in the image of v' , then at least one among the first or third quadrants are also in the image of v' .*

For an illustration of the statement of the Lemma take a look at Figure 18 and the point b_{42} there. The claim of the Lemma is that if the green region having b_{42} in its boundary is included in $Image(v')$, then one of the yellow regions next to b_{42} is also contained in this image.

Proof of Lemma 4.3.3. Consider a small segment $I \subset \pi(W'')$ that ends up at b_{ij} and is included in the closure of the fourth quadrant (the quadrants here are defined by the vertical and horizontal lines in Figure 18). We have $I \subset Image(v')$. Let $x \in I$. If x is the image of a point $z \in Int(S_r)$, then, by the open mapping theorem, the image of v' also intersects the third quadrant which implies our claim. Thus it is sufficient to consider the case when all the points of I are in the image of boundary points of S_r . The only boundary component that is mapped to W'' is C_{k+1} so that $I \subset v'(C_{k+1})$. Moreover, as b_{ij} is not the asymptotic image of the ends of C_{k+1} , it follows that $b_{ij} \in v'(C_{k+1})$. Let $z \in C_{k+1}$ so that $v'(z) = b_{ij}$. As shown at step 2, v' is holomorphic outside of \widehat{K} and thus, in particular, around b_{ij} . Given that (around b_{ij}) $v'(C_{k+1})$ is contained in the vertical line through b_{ij} and, due to the bottleneck structure around b_{ij} , the open mapping theorem implies that $Image(v')$ intersects the region of G corresponding to the first quadrant and ends the proof of the lemma. \square

We return to the proof of the proposition and we recall $v'(m) = b_{\alpha\beta}$, $v'(e) = b_{rs}$. Assume that $r > \alpha$. As m is an entry point, for orientation reasons, $Image(v')$ has to contain at least one of the first or third quadrants at $b_{\alpha\beta}$. In both cases, the upper left corner of the respective quadrant, that we denote by $b_{i_1j_1}$, is so that $i_1 \leq \alpha$. Thus Lemma 4.3.3 can be applied to $b_{i_1j_1}$ and it implies that the first or third quadrant at $b_{i_1j_1}$ is contained in $Image(v')$. Let $b_{i_2j_2}$ be the upper left corner of the respective quadrant. We have $i_2 \leq i_1$. This process can be pursued recursively, thus getting a sequence of points $b_{i_1j_1}, b_{i_2j_2}, \dots$ and associated quadrants $\subset Image(v')$ by picking at each step the upper left corner of a quadrant obtained from Lemma 4.3.3 applied to the previous point in the sequence. This process continues till one the quadrants in question is an unbounded region. But this contradicts the fact that the image of v' can not intersect such a region. See Figure 18 for an illustration of this argument. \square

4.4. Disjunction via Dehn twists. This subsection is purely geometric in nature and is of independent interest. Monotonicity assumptions are not required in this part. The main purpose here is to show that certain Dehn twists of a cobordism are Hamiltonian isotopic to remote cobordisms and therefore can be decomposed by means of Proposition 4.3.1. The idea

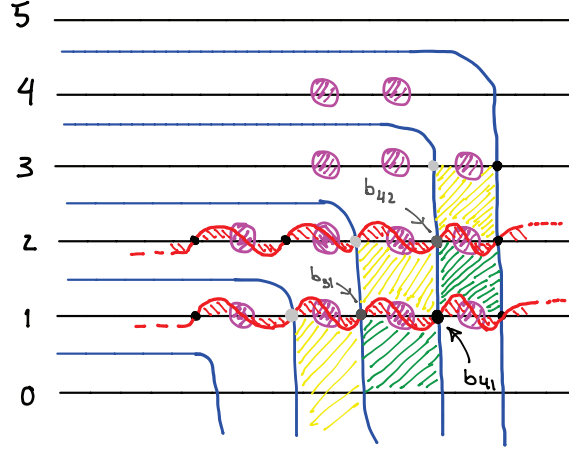


FIGURE 18. We take here $s \geq 5$ and in blue are the projections of the ends of W'' . Assume $v'(m) = b_{41}$ and suppose $v'(e) = b_{rs}$ with $r \geq 4$; v' exits b_{41} through one of the green regions which is therefore included in $\text{Image}(v')$; Lemma 4.3.3 applied to b_{42} and b_{41} shows that one of the yellow regions $\subset \text{Image}(v')$; by applying again Lemma 4.3.3 to one of the upper left corners of the yellow regions - in light gray - we get that an unbounded region of G is contained in $\text{Image}(v')$. Thus, we reach a contradiction in three steps.

is the following. Given a cobordisms $V \subset E$, we first add specific singularities to E (with critical values in the lower half plane) so that we can join each initial singularity x_i of E to one of the “new” ones, x'_i , by a matching cycle S_i . We then show that, with appropriate choices for the matching cycles and the other elements of the construction, the iterated Dehn twist $\tau_{S_m} \circ \dots \circ \tau_{S_i} \circ \dots \circ \tau_{S_1}$ transforms V into a remote cobordism V' .

4.4.1. *The case of a single singularity.* We start with the core of the geometric argument. This appears in the case of a fibration with a single singularity.

Fix $S \subset M$, a framed (or parametrized) Lagrangian sphere. We use Seidel’s terminology here [Sei2, Sei3] so that this means S is Lagrangian and that we fix a parametrization $e : S^n \rightarrow S$. Consider a Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$ which is tame outside $U \subset \mathbb{R} \times [\frac{1}{4}, +\infty) \subset \mathbb{C}$ and with a single singularity x_1 so that the vanishing cycle corresponding to x_1 coincides with S . (Note that since there is only one singularity here there is a canonical hamiltonian isotopy class of vanishing cycles in the fibers over $\mathbb{C} \setminus U$.) We will assume that the singularity has critical value $v_1 = (1, \frac{3}{2})$. Fix also a negatively ended cobordism $V \subset E$ with ends L_1, L_2, \dots, L_s .

For the construction described below it is useful to refer to Figure 19 (which contains also details that will be relevant only later on). We will make use of an auxiliary Lefschetz fibration $\hat{\pi} : \hat{E} \rightarrow \mathbb{C}$ that coincides with E over the upper half plane and that has an additional critical

point x'_1 with corresponding critical value $v'_1 = (-1, -\frac{3}{2})$ and a matching cycle $\hat{S}_\gamma \subset \hat{E}$ that projects onto \mathbb{C} to a path joining v'_1 to v_1 . More precisely, \hat{E} has the following properties. The fibration \hat{E} is tame outside a set \hat{U} (as pictured in Figure 19), $\hat{U} \subset (-\infty, a_{\hat{U}}] \times [-K, +\infty)$. Moreover, let D be a disk around v'_1 that is included in the lower half plane but is not completely included in \hat{U} . Let $v_0 \in \partial D \setminus \hat{U}$. Fix also a path γ that joins v_1 to v_0 . Denote by T_γ the thimble originating at x_1 and whose planar projection is γ . The boundary of T_γ is identified to the vanishing cycle S and, as subset in $\pi^{-1}(v_0)$, we denote it by S_0 . The fibration $\hat{\pi} : \hat{E} \rightarrow \mathbb{C}$ is such that it admits the sphere S_0 as vanishing cycle also relative to the singularity x'_1 . If we extend the curve γ to a curve (that we will continue to denote by γ) that joins v_1 to v'_1 this is covered by a matching cycle $\hat{S}_\gamma \subset \hat{E}$. Given that E is trivial over the lower half-plane, the construction of \hat{E} follows directly from the constructions in §16, [Sei3].

For further use, we now fix another thimble T originating at x_1 and whose projection is the vertical half-line $\{1\} \times [\frac{3}{2}, \infty)$.

Proposition 4.4.1. *There exists a curve γ , depending on V , and a framed Lagrangian sphere S' in \hat{E} , hamiltonian isotopic to the matching sphere \hat{S}_γ so that the Lagrangian $V' = \tau_{S'}V$ is disjoint from T and the intersection $V' \cap S'$ is contained in D .*

Proof. We start the proof by recalling the definition of the Dehn twist [Arn] following the conventions in [Sei2]. We begin with the model Dehn twist. This construction is standard in the subject but as we need the explicit definition in the following we will provide some details here. Let g be the standard round metric on S^n and for $0 < \lambda$ denote by $D_\lambda^* S^n \subset T^* S^n$ the disk bundle consisting of cotangent vectors of norm $\leq \lambda$. We have identified here $T^* S^n$ with TS^n via the metric g . Our conventions are such that the symplectic form on the cotangent bundle $T^* S^n$ is $dp \wedge dq$ where q is the “base” coordinate and p is the coordinate along the fiber.

Denote by $\psi_t : D_\lambda^* S^n \setminus 0_{S^n} \rightarrow D_\lambda^* S^n \setminus 0_{S^n}$ the *normalized* geodesic flow corresponding to g , defined on the complement of the zero-section. With our conventions this flow is the Hamiltonian flow of the function $H(p, q) = |p|$.

Denote by $\sigma : S^n \rightarrow S^n$ the antipodal map. Note that ψ_π extends to the zero-section by σ .

Given $0 < \lambda$, pick a smooth function $\rho_\lambda : \mathbb{R} \rightarrow \mathbb{R}$ with the following properties:

- (1) $\rho(t) + \rho(-t) = 1$ for every $|t| \leq \delta$ for some $0 < \delta < \lambda$.
- (2) $\text{supp}(\rho) \subset (-\lambda, \lambda)$; $\rho(t) \geq 0$, $\forall t > 0$.

Note that we have $\rho(0) = \frac{1}{2}$.

With the above at hand we define the model Dehn twist $\tau_{S^n} : D_\lambda^* S^n \longrightarrow D_\lambda^* S^n$ by the formula

$$(34) \quad \tau_{S^n}(x) = \begin{cases} \psi_{2\pi\rho(\|x\|)}^g(x), & x \in T_{\leq\lambda}^* S^n \setminus 0_{S^n}; \\ \sigma(x), & x \in 0_{S^n}. \end{cases}$$

Note that τ_{S^n} is the identity near the boundary of $D_\lambda^* S^n$.

Now let N be a symplectic manifold and $f : S^n \longrightarrow N$ a Lagrangian embedding of the n -sphere. Denote by $S = f(S^n) \subset N$ its image. By the Darboux-Weinstein theorem there exists a neighborhood $U(S) \subset N$ of S , a $\lambda > 0$, and a symplectic diffeomorphism $i : D_\lambda^* S^n \longrightarrow U(S)$ that maps 0_{S^n} to S via the map f . Define now the Dehn-twist along S , $\tau_S : N \longrightarrow N$, by setting $\tau_S = i \circ \tau_{S^n} \circ i^{-1}$ on the image of i and extend it as the identity to the rest of N . By the results of [Sei2] the diffeomorphism τ_S is symplectic and moreover, its symplectic isotopy class is independent of the choices of ρ and λ , but possibly not of the class of parametrization of the Lagrangian sphere $f : S^n \longrightarrow S$. The symplectomorphism τ_S is *the Dehn twist along S* .

Remark 4.4.2. In case S is a vanishing cycle in a Lefschetz fibration (associated to a path emanating from a critical value in the base of the fibration), S carries a canonical isotopy class of parametrizations (or framings) which we will often adopt implicitly. In that case τ_S is well defined up to symplectic isotopy without any further choices.

In the rest of the proof the place of N will be taken by \hat{E} and the role of S by the matching cycle \hat{S}_γ .

To start the actual proof we first assume that, after a possible Hamiltonian isotopy of V , T intersects V transversely in the points $p_1, \dots, p_k \in T$. All along the argument it is useful to refer to Figure 19.

Step 1: Choice of the curve γ . Recall that the fibration $\pi : E \rightarrow \mathbb{C}$ is tame outside the set $U \subset \mathbb{C}$ and the fibration $\hat{\pi} : \hat{E} \rightarrow \mathbb{C}$ is tame outside the larger set \hat{U} . We fix two neighborhoods $U(V) \subset U'(V)$ of V . We consider an auxiliary thimble \bar{T} whose projection on \mathbb{C} is as in Figure 19. In particular, \bar{T} coincides with T inside $U(V)$ as well as outside of $U'(V)$ and $\pi^{-1}(\mathbb{C} \setminus \hat{U}) \cap \bar{T} \neq \emptyset$ but $\pi^{-1}(\mathbb{C} \setminus \hat{U}) \cap \bar{T} \cap U(V) = \emptyset$. We notice that \bar{T} is hamiltonian isotopic to T by an isotopy supported away from $U(V) \cup \pi^{-1}(\mathbb{R} \times (-\infty, 0])$ (\bar{T} and T are Lagrangian isotopic and it is easy to check that this isotopy is exact).

Denote by $\bar{\eta} = \pi(\bar{T})$. We assume that, as in Figure 19, $\bar{\eta}$ can be written as the union of three closed connected sub-segments $\bar{\eta} = \bar{\eta}' \cup \bar{\eta}'' \cup \bar{\eta}'''$ so that $\bar{\eta}' \cup \bar{\eta}'''$ is the closure of $\hat{U} \cap \bar{\eta}$. Thus, the interior of $\bar{\eta}''$ is disjoint from \hat{U} . We also assume to fix that $\bar{\eta}'' \subset [1, \infty) \times [1, \infty)$. Consider a point e_0 inside the segment $\bar{\eta}''$ so that $\bar{\eta}'' = \bar{\eta}_1'' \cup \bar{\eta}_2''$ with $\bar{\eta}_1''$ and $\bar{\eta}_2''$ the closures

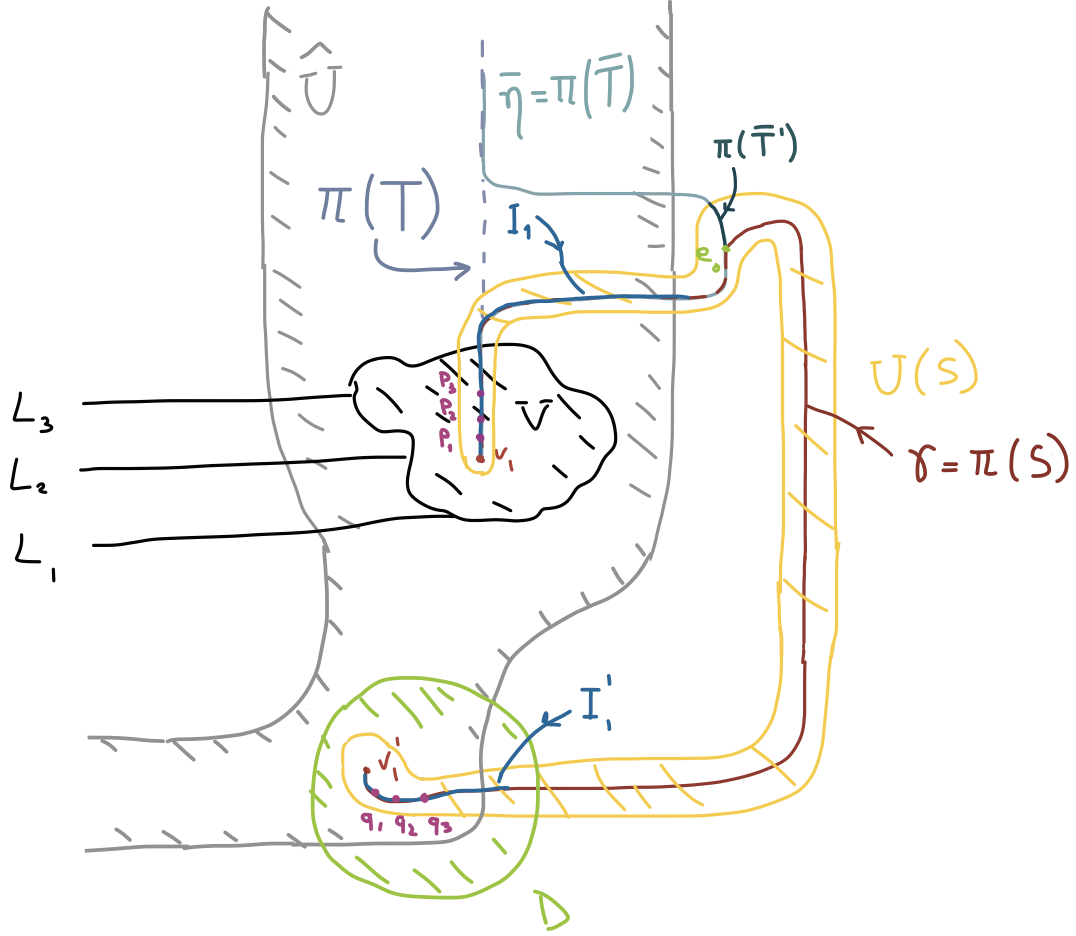


FIGURE 19. The Lefschetz fibration $\hat{\pi} : \hat{E} \rightarrow \mathbb{C}$ coincides with E over the upper semi-plane; $\hat{\pi}$ has two singularities of critical values v_1 and v'_1 and is symplectically trivial outside of \hat{U} . Are pictured (in projection on \mathbb{C}): the “straight” vertical thimble T and its deformation \bar{T} ; the matching cycle S that coincides with \bar{T} from v_1 to e_0 ; the disk D ; $S \cap V = \{p_1, p_2, p_3\}$; $q_i = \sigma(p_i)$ (where σ is the antipodal map); the neighborhood $U(S)$ where is supported τ_S ; the portion \bar{T}' of \bar{T} that differs from S and is included in $U(S)$; the projections I_1, I'_1 of two disks K_1, K'_1 in S around the two singularities of $\hat{\pi}$ so that $S_0 = S \setminus (K_1 \cup K'_1)$ lies inside a trivial symplectic fibration. Notice that the domain \hat{U} is generally unbounded along some additional directions compared to the domain outside which E is tame. This is required so that the fibration \hat{E} , that agrees with E over the upper half plane, has additional singularities compared to E . Our choice is for this unbounded direction to be in the lower left corner, as in the picture.

of the two sub-segments given by $\bar{\eta}'' \setminus \{e_0\}$ with e_0 being the end-point of $\bar{\eta}_1''$ and the starting point of $\bar{\eta}_2''$. We now pick the curve $\gamma \subset \mathbb{C}$ that joins v_1 to v_1' so that γ can be written as a union of two connected, closed parts $\gamma = \gamma_1 \cup \gamma_2$ so that γ_1 originates in v_1 and coincides with $\bar{\eta}' \cup \bar{\eta}_1''$, γ_2 is disjoint from $U(V)$, it intersects $\bar{\eta}$ only in e_0 , it ends in v_1' and $\gamma_2 \setminus D \subset \mathbb{C} \setminus \hat{U}$. Clearly, e_0 is a point where $\bar{\eta}$ and γ are tangent and after this point γ is to the “right” of $\bar{\eta}$ and is included in $\mathbb{C} \setminus \hat{U}$ till (and including) the moment it reaches D .

Notice that if we show that:

$$(35) \quad \tau_{\hat{S}_\gamma} V \cap \bar{T} = \emptyset \text{ and } \tau_{\hat{S}_\gamma} V \cap \hat{S}_\gamma \subset D$$

then by using the Hamiltonian isotopy ψ that carries \bar{T} to T and such that $\psi(V) = V$, we deduce that there is a Lagrangian sphere $S' = \psi(\hat{S}_\gamma)$ so that $\tau_{S'} V$ is disjoint from T and $\tau_{S'} V \cap S' \subset D$. For this argument, $\tau_{S'}$ is defined by using the choice of framing so that $\tau_{S'}^{-1} = \psi \circ \tau_{\hat{S}_\gamma}^{-1} \circ \psi^{-1}$ (hence $\tau_{S'}^{-1}(V) = \psi \circ \tau_{\hat{S}_\gamma}^{-1}(V)$). In short, it remains to show (35).

Step 2: *Other choices involved in the definition of the twist.* From now on, to simplify notation, we put $S = \hat{S}_\gamma$. We first choose a small Weinstein neighborhood $U(S)$ of S . The Dehn twist τ_S will be supported inside this neighborhood. We notice, by construction, that $\{p_1, \dots, p_k\} = T \cap V = \bar{T} \cap V = S \cap V$. We may assume that $V \cap U(S)$ is a union of small disks $D_i \subset V$ centered at p_i which, for convenience, we may assume are included in the fiber of T^*S through p_i under the identification of $U(S)$ with a disk bundle of T^*S . Further, we denote by \bar{T}' the closure of $(\bar{T} \setminus S) \cap U(S)$. We now consider a disk $K_1 \subset S$ centered at x_1 so that $U(V) \cap S \subset K_1$. Similarly we also consider a disk $K_1' \subset S$ centered at x_1' . We assume that both K_1 and K_1' are preimages of segments I_1 and I_1' contained in γ and we suppose that the two disks are so that $\gamma_0 = \gamma \setminus (I_1 \cup I_1') \subset \mathbb{C} \setminus \hat{U}$, $e_0 \in \gamma_0$ and $I_1' \subset D$. We further pick $U(S)$, K_1 and K_1' so that \bar{T}' is disjoint from both K_1 and K_1' . We consider the curve oriented so that it starts at v_1 and ends at v_1' .

The boundary of K_1 is a Lagrangian sphere $A \subset (M, \omega)$ and the boundary of K_1' is the same sphere transported to the end of γ_0 (parallel transport is trivial along γ_0 because $\hat{\pi}$ is symplectically trivial outside \hat{U}). We denote the sphere that appears as boundary of K_1' by A' . The region $S_0 = S \setminus \text{Int}(K_1 \cup K_1')$ is diffeomorphic to a cylinder $C = [-a, a] \times A$. We think about this cylinder so that $\{-a\} \times A$ corresponds to the boundary of K_1 and $\{a\} \times A$ corresponds to the boundary of K_1' .

Denote by $U(S_0)$ the restriction of the neighborhood $U(S)$ (identified with a disk bundle in T^*S) to S_0 . We assume $U(S)$ small enough so that $\pi(U(S_0)) \subset \mathbb{C} \setminus \hat{U}$. As $\hat{\pi}$ is trivial over $U(S_0)$, by possibly reducing $U(S)$ further, we obtain the existence of a symplectomorphism:

$$k : D_r T^*[-a, a] \times D_{r'} T^* A \rightarrow U(S_0) \approx D_s T^* S_0 \subset \hat{E}.$$

After picking a appropriately, this symplectomorphism can be made also compatible with the almost complex structures involved so that $\pi' = \hat{\pi} \circ k$ is holomorphic with respect to the split standard complex structure in the domain and the standard complex structure in \mathbb{C} .

Step 3: *The parametrization of S .* This step consists in picking a particular framing of S so that the associated Dehn twist τ_S can be tracked explicitly. To simplify slightly notation we assume $a = 1 - \delta$ with δ very small.

We fix a diffeomorphism $\varphi : S^n \rightarrow A$ in the isotopy class as explained at point (2) of Remark 4.4.2. Let $h : S^{n+1} \rightarrow \mathbb{R}$ be the height function defined on the standard round sphere in \mathbb{R}^{n+2} and let $S_\delta = h^{-1}([-a, a])$. We now pick a parametrization $\alpha : S^{n+1} \rightarrow S$ so that the restriction of this parametrization to S_δ is a diffeomorphism $\alpha_0 = \alpha|_{S_\delta} : S_\delta \rightarrow C$ with the property that for each $t \in [-a, a]$, $\alpha|_{h^{-1}(t)} : h^{-1}(t) \rightarrow \{t\} \times A \subset C$ is a rescaling of φ , and so that $h(\alpha^{-1}(x_1)) = -1$, $h(\alpha^{-1}(x'_1)) = 1$ (recall that $x_1, x'_1 \in \hat{E}$ are the critical points of π lying over v_1, v'_1 respectively). Clearly, α_0 extends to a symplectic diffeomorphism $\bar{\alpha}_0 : T^*S_\delta \rightarrow T^*C$ so that $T^*h^{-1}(t)$ is mapped by a symplectomorphism to $\{t\} \times T^*A$. Basically, we are parametrizing here the “flat” cylinder C (which is identified with S_0) by the “round” cylinder S_δ and we then extend this parametrization as symplectomorphisms at the level of the cotangent bundles. All the parametrizations involved identify level sets of the height function on S_δ to slices of the cylinder C .

We denote by $\sigma : S \rightarrow S$ the antipodal map defined using this parametrization. This means, in particular, that the points $q_i = \sigma(p_i)$ are contained in D (the disk appearing in the statement of the proposition). It is easy to see, as for instance in §1.2 [Sei2], with an appropriate choice of function ρ in the definition of the Dehn twist (which we have assumed here) the intersection $\tau_S V \cap S$ is transverse and consists precisely of the antipodal of the intersection $S \cap V$. Thus, $\tau_S V \cap S = \{q_1, \dots, q_k\} \subset D$ as claimed in the second part of (35). It remains to show the main part of the claim: $\tau_S V \cap \bar{T} = \emptyset$. As $\tau_S V \cap S = \{q_1, \dots, q_k\}$, the Dehn twist τ_S is supported inside $U(S)$ and given that \bar{T} and S coincide along the segment of γ that starts at v_1 and ends at e_0 it follows that

$$(36) \quad \tau_S V \cap \bar{T} = \tau_S V \cap \bar{T}' = \tau_S(V \cap \tau_S^{-1}(\bar{T}'))$$

Thus, to conclude the proof, it is enough to show $\tau_S^{-1}(\bar{T}') \cap V = \emptyset$.

Step 4: *Showing $\tau_S^{-1}(\bar{T}') \cap V = \emptyset$.* By possibly adjusting the neighborhood $U(S)$ we may assume that U can be written as $U(S) = (k \circ \bar{\alpha}_0)(U(S^{n+1}))$ for some neighborhood $U(S^{n+1})$ of the zero section inside T^*S_δ . Let $\tilde{T}' = (k \circ \bar{\alpha}_0)^{-1}(\bar{T}')$. We denote by $U(S_\delta)$ the corresponding neighborhood of S_δ (so that $U(S_\delta)$ is the preimage of $U(S_0)$) and we let \tilde{K}_1 be the cap $\tilde{K}_1 = h^{-1}(-1, -1 + \delta] = (k \circ \bar{\alpha}_0)^{-1}(K_1)$. Further, we let $U(\tilde{K}_1)$ be the restriction of $U(S^{n+1})$ over

\tilde{K}_1 . Clearly $\tilde{T}' \subset U(S_\delta)$, and to show the claim it is enough to notice that $\tau_{S^{n+1}}^{-1} \tilde{T}' \cap U(\tilde{K}_1) = \emptyset$ where now $\tau_{S^{n+1}}$ is the standard model for the Dehn twist.

Let $(x, v) \in \tilde{T}' \subset T^*S_\delta$ with $v \in T_x^*S^{n+1}$, $v \neq 0$. We now notice that the condition that \tilde{T}' is to the “left” of S in Figure 19 translates to the fact that

$$(37) \quad \langle v, J\nabla h(x) \rangle > 0 .$$

Here J is an almost complex structure on T^*S_δ with respect to which, as at Step 2, the map $\pi' = \hat{\pi} \circ k$ is holomorphic. This follows from the same inequality that is valid for the planar projection of \tilde{T}' relative to γ_0 . Equation (37) implies that the geodesic flow with origin (x, v) has its vertical component pointing in the direction of $-\nabla h$ (because if $\langle v, w \rangle > 0$, then the geodesic associated to v points in the direction of Jw). Thus, the inverse of the geodesic flow points in the direction of ∇h and therefore away from \tilde{K}_1 . As a consequence, it is easy to see that the orbit $\phi_t^g(x, v)$ for $-\pi \leq t \leq 0$ does not intersect $U(\tilde{K}_1)$ and, as a consequence, $\tau_S^{-1}(\tilde{T}') \cap V = \emptyset$ - see also Figure 20. \square

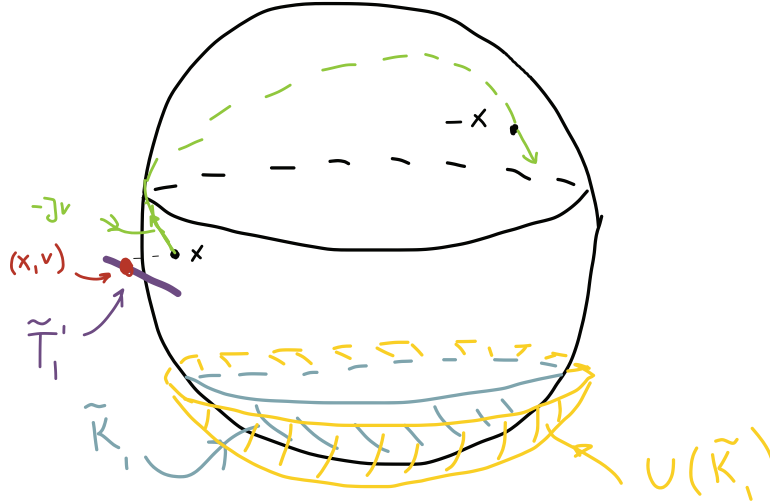


FIGURE 20. The cap $\tilde{K}_1 \subset S^{n+1}$ the set \tilde{T}'_1 containing the point (x, v) together with the geodesic starting from x in the direction of $-Jv$ and ending at $-x$.

Corollary 4.4.3. *With the notation in Proposition 4.4.1 the cobordism $\tau_{S'}V$ is hamiltonian isotopic - via an isotopy with compact support - to a cobordism that is remote relative to E .*

Proof. We already know from Proposition 4.4.1 that $V' = \tau_{S'}V$ is disjoint from T . Consider an Ω -compatible almost complex structure J on E with the additional property that $\pi : E \rightarrow \mathbb{C}$ is J -holomorphic. It is well known that the function $Im(\pi) : E \rightarrow \mathbb{R}$ defines a Morse function on E whose negative gradient flow ξ (with respect to the metric induced

by (Ω, J) is also Hamiltonian. Moreover ξ has the thimble T as a stable manifold. Write $\xi = X^H$ with $H : E \rightarrow \mathbb{R}$. Now consider a smooth function $\eta : \mathbb{C} \rightarrow \mathbb{R}$ so that $\eta(z) = 1$ if $z \in [-a_U - 1, a_U + 1] \times [-\frac{1}{4}, +\infty)$ and $\eta(z) = 0$ if $z \in ((-\infty, -a_U - 2] \times \mathbb{R}) \cup ([-a_U - 2, a_U + 2] \times (-\infty, -\frac{1}{2}]) \cup ([a_U + 2, \infty) \times \mathbb{R})$. Let ξ' be the Hamiltonian flow of the function $(\eta \circ \pi)H$ defined on \hat{E} . It is easy to see that, after sufficient time, the flow ξ' isotopes V' to a new cobordism V'' that is included in $\hat{\pi}^{-1}(\mathbb{R} \times (-\infty, 0] \times \mathbb{R} \cup Q_U^-)$. Therefore, V'' is remote relative to E . Moreover, as the ends of V' are not moved by this isotopy, it is easy to see that, by a further truncation of ξ' , V'' is hamiltonian isotopic to V' through a compactly supported isotopy. \square

4.4.2. Multiple singularities. Consider a Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$ as in §4.1, thus possibly with more than one singularity.

We fix $V \in \mathcal{Ob}(\mathcal{Fuk}^*(E))$, $V : \emptyset \rightsquigarrow (L_1, \dots, L_s)$. The purpose of this subsection is to describe an extension of Proposition 4.4.1 and Corollary 4.4.3 to the case of multiple singularities.

We will consider a fibration $\hat{\pi} : \hat{E} \rightarrow \mathbb{C}$ that extends E and has one more singularity x'_i for each singular point x_i , $1 \leq i \leq m$, of π so that the vanishing cycles of x_i and x'_i can be related by matching cycles \hat{S}_i that are the analogues of the matching cycle \hat{S}_γ from Proposition 4.4.1. The specific positioning of the corresponding critical values v'_i in the plane \mathbb{C} is important as is as in Figure 21. We then obtain Lagrangian spheres, S'_i that are hamiltonian isotopic to \hat{S}_i (as in Figure 21) and we then consider the image of V under the iterated Dehn twist

$$V' = \tau_{\hat{S}_m} \circ \tau_{\hat{S}_{m-1}} \circ \dots \circ \tau_{\hat{S}_1}(V)$$

inside \hat{E} as well as the following Hamiltonian isotopic copy of it $V'' = \tau_{S'_m} \circ \tau_{S'_{m-1}} \circ \dots \circ \tau_{S'_1}(V)$ obtained by applying an iterated Dehn twist along the Lagrangian spheres S'_j which are Hamiltonian isotopic to the \hat{S}_j 's.

Let \mathcal{T}_i be the vertical thimble with origin the critical point x_i and projecting to the vertical half-line $\{i\} \times [\frac{3}{2}, \infty)$. The thimbles \mathcal{T}_i generalize the thimble T considered earlier (just before Proposition 4.4.1) in the context of one singularity to the case of multiple singularities. We denote them by \mathcal{T}_i (this avoids confusion with the thimbles T_i that are horizontal at infinity and are associated to the curves t_i , see Figure 10).

Corollary 4.4.4. *It is possible to construct \hat{E} and the Lagrangian spheres S'_i so that the cobordism V'' is disjoint from all the thimbles \mathcal{T}_i . As a consequence, there exists a horizontal Hamiltonian isotopy ϕ so that the cobordism $\phi(V'') \subset \hat{E}$ is remote relative to E . In particular, in $D\mathcal{Fuk}^*(E)$, there exists a cone decomposition:*

$$V'_E \cong (\gamma_s \times L_s \rightarrow \gamma_{s-1} \times L_{s-1} \rightarrow \dots \rightarrow \gamma_2 \times L_2) .$$

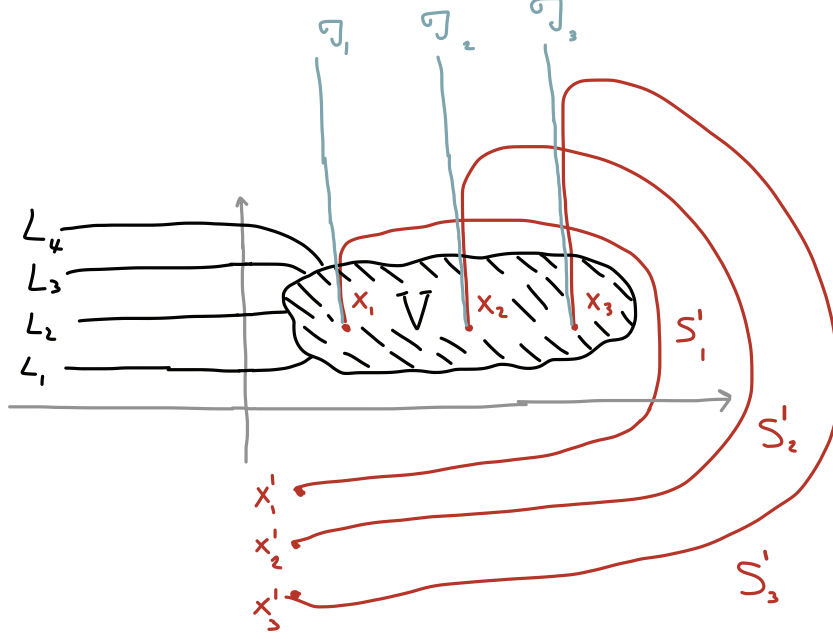


FIGURE 21. The cobordism $V : \emptyset \rightsquigarrow (L_1, L_2, L_3, L_4)$, the Lagrangian spheres S'_1, S'_2, S'_3 together with the vertical thimbles $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ so that $V'' = \tau_{S'_m} \circ \tau_{S'_{m-1}} \circ \cdots \circ \tau_{S'_1}(V)$ is disjoint from the \mathcal{T}_i 's.

Proof. The first part of the proof is to construct iteratively fibrations $\hat{\pi}_i : \hat{E}_i \rightarrow \mathbb{C}$ with $\hat{E}_0 = E$ and with the final fibration $\hat{E} = \hat{E}_m$ so that \hat{E}_{i+1} extends \hat{E}_i and has one more singularity, x'_{i+1} , compared to \hat{E}_i . At each step we also construct the matching cycles \hat{S}_i joining x_i to x'_i and their Hamiltonian isotopic images S'_i so that the relevant properties are satisfied. Here are more details on the induction step. Assume that \hat{E}_k has already been constructed together with the matching cycles \hat{S}_i and their hamiltonian isotopic copies S'_i , $1 \leq i \leq k$ so that $V''_k = \tau_{S'_k} \circ \tau_{S'_{k-1}} \circ \cdots \circ \tau_{S'_1}(V)$ is disjoint from \mathcal{T}_i , $1 \leq i \leq k$. We now consider the cobordism V''_k and the vertical thimble \mathcal{T}_{k+1} and we apply to them the construction described in the proof of Proposition 4.4.1. This produces first a new fibration \hat{E}_{k+1} that has an additional singularity denoted now by x'_{k+1} . Here, the only difference with respect to the construction of \hat{E} in Proposition 4.4.1 is that the coordinates of the critical value v'_{k+1} associated to x'_{k+1} is now $(-1, -k - \frac{3}{2})$ and the set \hat{U} , outside which \hat{E}_{k+1} is tame, is extended appropriately inside the third-quadrant. Further, just as in the proof of Proposition 4.4.1 we can construct the deformed thimble $\bar{\mathcal{T}}_{k+1}$ as well as the matching cycle \hat{S}_γ so that \hat{S}_γ coincides with $\bar{\mathcal{T}}_{k+1}$ over a certain sub-segment of γ . Two important points should be made here: first, the place of V in the proof of Proposition 4.4.1 is taken here by V''_{k+1} ; second \mathcal{T}_{k+1} as well as $\bar{\mathcal{T}}_{k+1}$ and \hat{S}_γ are all disjoint from \mathcal{T}_i for $i \leq k$. Now, again as in the proof of Proposition 4.4.1, we obtain that there exists a hamiltonian isotopy ψ_{k+1} supported outside a neighborhood of V''_{k+1} so that

$S'_{k+1} = \psi_{k+1}(\hat{S}_\gamma)$ has the property that $V''_{k+1} = \tau_{S'_{k+1}} V''_k$ is disjoint from \mathcal{T}_{k+1} . One additional point appears here: it is easy to see that the isotopy ψ_{k+1} can be assumed to leave fixed \mathcal{T}_i for $i \leq k$. By defining V''_{k+1} by using a sufficiently small neighborhood $U(S'_{k+1})$ of S'_{k+1} so that $U(S'_{k+1}) \cap \mathcal{T}_i = \emptyset$ for all $i \leq k$, we also deduce $V''_{k+1} \cap \mathcal{T}_i = \emptyset$ $1 \leq i \leq k$ and the induction step is completed.

We now put $V'' = V''_m$ and we know that V'' is disjoint from all the thimbles \mathcal{T}_i . Constructing the horizontal isotopy that transforms V'' into a cobordism V''' remote relative to E is a simple exercise by, possibly, iterating the construction in Corollary 4.4.3.

Finally, the cone-decomposition in the statement follows by applying to V''' Proposition 4.3.1. \square

The following proposition establishes monotonicity properties for \hat{E} that will be used later on in §4.6 when proving Theorems 4.2.1 and A.

Proposition 4.4.5. *If the Lefschetz fibration $E \rightarrow \mathbb{C}$ is strongly monotone (see Definition 3.2.1) then the extended fibration $\hat{E} \rightarrow \mathbb{C}$ is strongly monotone too and has the same monotonicity class $*$. The matching spheres $\hat{S}_j \subset \hat{E}$ are monotone of class $*$ and if the cobordism $V \subset E$ is monotone of class $*$ then it continues to be monotone of the same class when viewed as a cobordism in \hat{E} .*

Proof. Denote by M the generic fiber of E . Assume first that $\dim_{\mathbb{R}} M \geq 4$. By Remark 3.2.2 M is monotone. Denote for every $1 \leq j \leq m$ by λ_j the path connecting x_j to x'_j over which the matching cycle \hat{S}_j was constructed, as in Figure 21. Pick a point p_j on λ_j in such a way that all the points p_1, \dots, p_m are in the upper half-plane and all of them lie in one of the domain where \hat{E} is tame. Divide each of the λ_j into two parts: λ_j^+ going from x_j to p_j and λ_j^- that goes (in the opposite orientation to λ_j) from x'_j to p_j . Since \hat{S}_j is a matching cycle, the two vanishing spheres in $E_{p_j} = \pi^{-1}(p_j)$ associated to the paths λ_j^+ and λ_j^- coincide. It follows that if Case (ii) in Definition 3.2.1 is applicable then it is satisfied also for the fibration \hat{E} . This proves that \hat{E} is strongly monotone under the assumption that $\dim_{\mathbb{R}} M \geq 4$. It is not hard to see that its monotonicity class $*$ is the same as the one of E . That V remains monotone when viewed in \hat{E} follows easily from the fact that when $\dim_{\mathbb{R}} M \geq 4$ the map induced by the inclusion $\pi_2(E, V) \rightarrow \pi_2(\hat{E}, V)$ is surjective.

The statement about the matching spheres will be proved below, at the present proof, as it does not require any assumptions on the dimension of M .

We now turn to the case $\dim_{\mathbb{R}} M = 2$. Recall that in this case strong monotonicity assumes that E itself is a monotone manifold. We will first determine the homotopy type of E and that of \hat{E} . Consider the complement (in \mathbb{C}) of the union of curves $\cup_{j=1}^m \lambda_j$. This has several unbounded connected components and several bounded ones (unless $m = 1, 2$, when there are only unbounded ones). Denote by $\mathcal{B} \subset \mathbb{C}$ the closure of the union of the bounded components.

If $m = 1$ take \mathcal{B} to be just a point on λ_1 in the upper half-plane which is not x_1 or x'_1 and if $m = 2$ take $\mathcal{B} = \lambda_1 \cap \lambda_2$. Put $\hat{E}_{\lambda, \mathcal{B}} = \hat{\pi}^{-1}(\cup_{j=1}^m \lambda_j \cup \mathcal{B})$. Then the inclusion $\hat{E}_{\lambda, \mathcal{B}} \longrightarrow \hat{E}$ is a homotopy equivalence.

Denote by $l_j^+ \subset \lambda_j$ the part of λ_j that starts from x_j till the point where it enters \mathcal{B} , and by l_j^- the path starting at x'_j and goes along λ_j , with the reverse orientation, till the point it hits the domain \mathcal{B} . Put $E_{l^+, \mathcal{B}} = \pi^{-1}(\cup_{j=1}^m l_j^+ \cup \mathcal{B})$. The inclusion $E_{l^+, \mathcal{B}} \longrightarrow E$ is a homotopy equivalence too.

Consider now the following subspaces:

$$\hat{E}^0 = E|_{\mathcal{B}} \cup (\cup_{j=1}^m T_{l_j^+}) \cup (\cup_{j=1}^m T_{l_j^-}) \subset \hat{E}_{\lambda, \mathcal{B}}, \quad E^0 = E|_{\mathcal{B}} \cup (\cup_{j=1}^m T_{l_j^+}) \subset E_{l^+, \mathcal{B}},$$

where $T_{l_j^+}$ is the thimble associated to l_j^+ and similarly for $T_{l_j^-}$. Thus \hat{E}^0 is obtained from $E_{\mathcal{B}}$ by attaching to it m pairs of $(n+1)$ -dimensional balls by identifying their boundaries with vanishing spheres of some fibers of E over \mathcal{B} . The space E^0 has an analogous description, by using only the $T_{l_j^+}$'s. Note also that

$$\hat{E}^0 = E|_{\mathcal{B}} \cup (\cup_{j=1}^m \hat{S}_j).$$

By standard arguments from Morse theory the inclusions $\hat{E}^0 \longrightarrow \hat{E}_{\lambda, \mathcal{B}}$ and $E^0 \longrightarrow E_{l^+, \mathcal{B}}$ are homotopy equivalences.

We are now ready to show that \hat{E} is a monotone symplectic manifold. For a space X we denote by $H_2^S(X) = \text{image}(\pi_2(X) \longrightarrow H_2(X))$ the image of the Hurewicz homomorphism. Denote by $j : E^0 \longrightarrow \hat{E}^0$ the inclusion and by j_* its induced map on H_2^S . Since \mathcal{B} is contractible, it is easy to see that $H_2^S(\hat{E}^0)$ is generated by $\text{image}(j_*)$ together with the classes $[\hat{S}_1], \dots, [\hat{S}_m]$. As E is assumed to be monotone and \hat{S}_j are Lagrangian it readily follows that \hat{E} is monotone too.

The monotonicity of $V \subset \hat{E}$ can be proved by similar methods. For a pair of spaces $Y \subset X$ put $H_2^D(X, Y) = \text{image}(\pi_2(X, Y) \longrightarrow H_2(X, Y))$. A similar argument to the preceding one combined with the homotopy long exact sequence of the triple (\hat{E}, E, V) shows that $H_2^D(\hat{E}, V)$ is generated by $\text{image}(i_*)$ together with $[\hat{S}_1], \dots, [\hat{S}_m]$, where i_* is the map induced by the inclusion $(E, V) \rightarrow (\hat{E}, V)$ and the $[S_j]$'s are viewed as elements of $H_2^D(\hat{E}, V)$ via the map $H_2^S \rightarrow H_2^D$. As before, since $[S_j]$ are Lagrangian it follows that $V \subset \hat{E}$ remains monotone. Moreover, it is easy to see that its monotonicity class $*$ remains unchanged.

Finally, we prove the statement about the matching spheres. The argument below works for M of arbitrary positive dimension. Let \hat{S} be one of the matching spheres \hat{S}_j . Since \hat{S} is simply connected (recall that $\dim M > 0$) and \hat{E} is monotone it follows that \hat{S} is monotone too. Moreover, if the monotonicity constant of E satisfies $\rho > 0$, then \hat{S} will have the same constant.

It remains to show that $d_{\hat{S}} = d_E$. Recall that $d_{\hat{S}}$ counts the number of pseudo-holomorphic disks in \hat{E} with boundary on \hat{S} that go through a given point in \hat{S} . Pick a point $p \in \hat{S}$ such that its projection $z = \pi(p)$ belongs to the upper half-plane and is in a region where both E and \hat{E} are tame. Denote by $\mathcal{U} \subset \mathbb{C}$ the domain over which \hat{E} is tame. Let \hat{J} be an almost complex structure on \hat{E} , compatible with the symplectic structure and such that $\pi : \hat{E} \rightarrow \mathbb{C}$ is \hat{J} -holomorphic above \mathcal{U} . Standard arguments show that the class of such almost complex structures contain regular ones and therefore one can calculate $d_{\hat{S}}$ using such a \hat{J} .

Let $u : (D, \partial D) \rightarrow (\hat{E}, \hat{S})$ be a \hat{J} -holomorphic disk with $u(\partial D) \ni p$. Let $v = \pi \circ u : D \rightarrow \mathbb{C}$ be the projection of u to \mathbb{C} . We claim that v is constant, hence the image of u is in the fiber $E_z \cong M$. To prove this, suppose by contradiction that v is not constant. We have $v(\partial D) \subset \lambda$, where $\lambda = \pi(\hat{S})$ is a curve (connecting two critical values x_j and x'_j of π). Note that v is holomorphic on $\mathcal{H} := v^{-1}(\mathcal{U})$. Let $\xi \in \partial D$ be a point such that $v(\xi) = z$. Clearly $\xi \in \mathcal{H}$. Without loss of generality we may assume that z is not a critical value of v (otherwise, move z slightly to a nearby point on λ which is still in the image of v and which is a regular value of v). By the open mapping theorem it is impossible for $v(D)$ to intersect the part of \mathcal{U} that is on the right-hand side of λ . Thus in a neighborhood of z , the image $v(D)$ must be on the left-hand side of λ . Since v is holomorphic near ξ it follows that when we go along ∂D counterclockwise through ξ , the image of v goes along λ in the upper direction. This holds for all points $\xi \in v^{-1}(z)$. But this is impossible since λ is not a closed curve, so there must be another point $\xi' \in \partial D$ with $v(\xi') = z$ and such that when we go counterclockwise along ∂D around ξ' the image of v goes in the lower direction of λ . A contradiction. This proves that v is constant, hence $\text{image } u \subset E_z$. We thus conclude that $d_{\hat{S}} = d_S$, where $S \subset E_z$ is the vanishing sphere corresponding to the matching sphere \hat{S} . It now easily follows that $d_{\hat{S}} = d_E$. \square

4.4.3. Dehn twist as multiple surgery. Here we give an interpretation of the action of a Dehn twist on Lagrangian submanifolds in terms of surgery. Fix $S^n \rightarrow S \subset M$, a parametrized Lagrangian sphere and let L be another Lagrangian submanifold of (M, ω) . It is known that if L and S intersect transversely and in a single point, then Lagrangian surgery at this point produces a Lagrangian $S \# L$ that is Hamiltonian isotopic to the Dehn twist $\tau_S L$ of L along S (see e.g. [Sei1, Tho]). (See [Pol] as well as [LS] for the definition of Lagrangian surgery, and see below for our conventions regarding the choice of handles in the surgery). Assume now that L is still transverse to S but that the number of intersection points $L \cap S$ is more than one. In this case too, one can express the Dehn twist $\tau_S(L)$ as a certain type of surgery. The construction goes as follows. Assume that $L \cap S = \{p_1, \dots, p_r\}$. Fix an additional point $p_0 \in S$ and a small neighborhood of it $V \subset S$.

- i. Consider r hamiltonian diffeomorphisms ϕ^j , $1 \leq j \leq r$ supported in a small Weinstein neighborhood of S , so that $S_j = \phi^j(S)$ is transverse to S and $S_j \cap S = \{p_j, p'_j\}$ for some additional point $p'_j \in V$.
- ii. Pick small disks $D_j^L \subset L$ centered at p_j and disks $D_j^{S_j} \subset S_j$ also centered at p_j as well as Lagrangian handles $H_j \subset M$ defined in a small neighborhood of p_j that join S_j to L so that $(L \setminus D_j^L) \cup (S_j \setminus D_j^{S_j}) \cup H_j$ is the usual Lagrangian surgery $L \# S_j$ between L and S_j at the point p_j (this is, in general, an immersed Lagrangian). Notice that there are two choices for Lagrangian surgery at each intersection point. The choice used here is the same at each point and is the one defined as follows (this is the same convention as in [BC2]). The sphere S is oriented hence so are the S_j 's. This induces a local orientation on L (even if L is not orientable) near each intersection point p_j in such a way that $T_{p_j} S_j \oplus T_{p_j} L$ gives the orientation of $T_{p_j} M$. We then symplectically identify a neighborhood of $p_j \in M$ with a neighborhood of 0 in \mathbb{R}^{2n} in such a way that $D_j^{S_j}$ is identified with a small disk around 0 in $\mathbb{R}^n \times \{0\}$ and D_j^L with a small disk around 0 in $\{0\} \times \mathbb{R}^n$, with the last two identifications being orientation preserving. The model Lagrangian handle is then defined to be $H_j = \cup_{t \in [-1, 1]} \gamma(t) S^{n-1} \in \mathbb{C}^n \cong \mathbb{R}^{2n}$, where $\gamma(t) : [-1, 1] \rightarrow \mathbb{C}$ is an appropriately chosen curve whose image is in the 2'nd quadrant and such that $\gamma(t) \in \mathbb{R}_{<0}$ for t close to -1 and $\gamma(t) \in i\mathbb{R}_{>0}$ for t close 1.
- iii. Define $S \#_r L$ by

$$(38) \quad S \#_r L = (\cup_j S_j \setminus D_j^{S_j}) \cup (L \setminus \cup_j D_j^L) \cup (\cup_j H_j) .$$

In other words $S \#_r L$ is obtained by performing simultaneously, for all $1 \leq j \leq r$, the one point surgery at p_j between S_j and L .

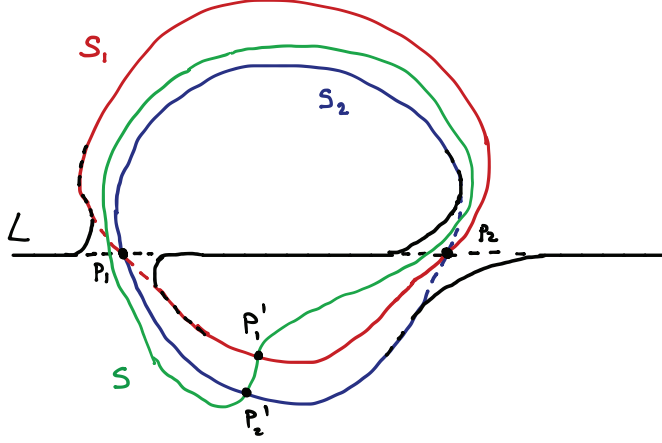


FIGURE 22. Dehn twist as multiple surgery for $n = 1$ assuming two intersection points p_1, p_2 between L and S .

Either by a direct argument - this is instructive to draw in dimension two as in Figure 22 - or by comparing this multiple surgery construction with the definition of $\tau_S L$, we see that there exist choices of $\phi^j, D_j^L, D_j^{S_j}, H_j$ so that:

- i. $S \#_r L$ is embedded and is Hamiltonian isotopic to $\tau_S L$.
- ii. $S \#_r L$ is transverse to S and it intersects S in the r points $p'_j \in V$, $1 \leq j \leq r$.
- iii. If both L and S are monotone of monotonicity constant ρ , then so is $S \#_r L$.

As explained above, the local model for surgery at a point requires an order among the two Lagrangians involved. By reversing the order for all the one-point surgeries, we obtain again a Lagrangian denoted now $L \#_r S$. This has properties similar to i,ii,iii above except that it is hamiltonian isotopic to $\tau_S^{-1} L$. From this perspective, Proposition 4.4.1 claims that, with appropriate choices of handles, we have $(S' \#_r V) \cap T = \emptyset$.

Remark 4.4.6. a. The “doubling” of singularities used in Proposition 4.4.1 first appeared in a somewhat different form and with a different purpose in the work of Seidel [Sei3]. From the perspective of our paper, the initial approach to the setting of Proposition 4.4.1 was to consider a thimble T' (inside E) that projects over the curve γ in Figure 19 and continues horizontally to $-\infty$. The idea was to disjoin V from T by a process of multiple surgery with multiple copies of T' , in other words to define $V' = T' \#_r V$ so that $V' \cap T = \emptyset$. Purely geometrically, this operation is possible. However, the problem in drawing algebraic conclusions from it is that the condition $V' \cap T = \emptyset$ turns out to force that the copies of T' used in the surgery are not cylindrical at infinity (alternatively, one can achieve cylindricity at infinity at the expense that the resulting manifold V' would no longer be embedded but only immersed, see also §6.3). As a consequence the machinery involving J -holomorphic curves can not be applied directly

to V' . On the other hand, by compactifying T' to the sphere S' - as described in the paper - this issue is no longer present. The price to pay is that we need to add singularities to the initial fibration E .

b. It is likely that Proposition 4.4.1 can be proven also along an approach closer to Seidel's constructions involving bifibrations. The basic idea along this line would be to construct the fibration \hat{E} by symmetrizing the restriction of the fibration E to the upper semi-plane by a rotation σ by 180° around the origin in \mathbb{C} . This gives rise to a specific matching cycle that projects to a segment joining the singular value v_1 to its "mirror" v'_1 . By restricting to a suitable disk D containing this segment, we see that the the Dehn twist around this vanishing cycle is identified to the rotation σ (Lemma 18.2 in [Sei3]). At the same time if V is assumed to be a Lagrangian without ends and included in D , then $\sigma(V)$ is remote. However, as V is in general more complicated this argument does not work directly and thus we gave a direct geometric proof.

4.5. A cobordism viewpoint on Seidel's exact triangle. In this section we present a new proof of Seidel's exact triangle [Sei2, Sei3]. This is the last essential ingredient for the proof of Theorem 4.2.1. Our proof is based on cobordism considerations and is valid in the monotone setting. We give full details not only for the sake of self-containedness but also in order to emphasize the reason why the Novikov ring \mathcal{A} is required in the proof of Theorem 4.2.1: this is precisely in establishing Seidel's exact triangle. Additionally, in the proof of Theorem 4.2.1 we need a variant of the exact triangle that applies to the case when the Lagrangian to which the Dehn twist is applied is itself a cobordism in the total space of a Lefschetz fibration and the proof is robust enough to cover this case with minimal adjustment.

Seidel's proof [Sei3] assumes an exact setting but his argument adapts to the monotone case too and also admits further generalizations as in [WW].

4.5.1. The exact triangle. We work, as in the rest of the paper, with coefficients in the universal Novikov ring \mathcal{A} over \mathbb{Z}_2 and with monotone Lagrangians assumed to be of class $*$. Floer complexes and Fukaya categories are ungraded.

Below we will have two versions of the Seidel's exact triangle. The first is for symplectic manifolds X (which are either closed or symplectically convex at infinity) and their compact Fukaya categories (i.e. the Fukaya categories whose objects are *closed* Lagrangian submanifolds). The second version is specially tailored to the situation when X is itself the total space of a Lefschetz fibration and the Fukaya category considered in X is that of negatively ended cobordisms in X . It is the second version that will be used in the proof of Theorem 4.2.1. We will later exhibit X as a fiber in a Lefschetz fibration denoted by \mathcal{E} . The choice of notation (\mathcal{E} and X) is intentional, in order to avoid confusion with the Lefschetz fibrations $E \rightarrow \mathbb{C}$ and their fibers M that appear in the rest of the paper.

Let (X^{2n+2}, ω) be a symplectic manifold which is either closed or symplectically convex at infinity. Throughout this section we add the assumption that $\dim_{\mathbb{R}} X \geq 4$. (The reason for this restriction will be explained in Remark 4.5.4 below.) Let S a parametrized Lagrangian sphere in X , i.e. a Lagrangian submanifold $S \subset X$ together with a diffeomorphism $i_S : S^{n+1} \rightarrow S$. Recall that we denote by $\tau_S : X \rightarrow X$ the Dehn twist associated to S . Assume further that $S \subset X$ is monotone and denote by $*$ its monotonicity class. Following the conventions of the paper, we write $\mathcal{Fuk}^*(X)$ for the Fukaya category of monotone closed Lagrangian submanifolds of X of monotonicity class $*$.

The following important result was proved by Seidel [Sei2] in the exact case. As mentioned above, we extend the result to the monotone case and provide an independent proof.

Proposition 4.5.1. *Let X, S be as above and let $Q \subset X$ be another monotone closed Lagrangian submanifold of monotonicity class $*$. In $D\mathcal{Fuk}^*(X)$ there is an exact triangle of the form:*

$$(39) \quad \begin{array}{ccc} \tau_S(Q) & \xrightarrow{\quad} & Q \\ & \nwarrow & \downarrow \\ & & S \otimes HF(S, Q) \end{array}$$

The proof of this result will occupy most of §4.5.3 below. We note that the maps appearing in this exact triangle will be identified along the proof, they coincide with the corresponding maps in Seidel's exact triangle.

Remark 4.5.2. If one restricts the objects in the Fukaya category of X to orientable Lagrangians, our proof should hold also with a \mathbb{Z}_2 -grading. Similarly, under more assumptions on the Lagrangians (and additional structures) the proof is expected to carry over with a \mathbb{Z} -grading as well as, if one assumes all Lagrangians to be endowed with spin structures, with coefficients in \mathbb{Z} .

4.5.2. Second version of the exact triangle: the case when X is a Lefschetz fibration. Here we assume that X is the total space of a tame Lefschetz fibration $\pi_X^{2n+2} : X \rightarrow \mathbb{C}$, $n \geq 1$, as defined in §2. (The assumption that X is symplectically convex at infinity is now dropped.) We denote by $\mathcal{Fuk}^*(X)$ the Fukaya category of X whose objects are negatively ended *Lagrangian cobordisms* in X of monotonicity class $*$ as defined in §3.3.

Proposition 4.5.3. *For X as above, let $S \subset X$ be a monotone Lagrangian sphere of class $*$ and let $Q \subset X$ be a monotone Lagrangian cobordism (possibly without ends) of the same monotonicity class. Then in $D\mathcal{Fuk}^*(X)$ there is an exact triangle as in (39).*

The proof is very similar to the proof of Proposition 4.5.1 (which is given in §4.5.3 below), the only difference being that now Q is allowed to be a cobordism rather than just a closed

Lagrangian (and similarly for the objects of $\mathcal{Fuk}^*(X)$). We explain the necessary modifications in §4.5.7 below.

4.5.3. *Outline of the proof of Proposition 4.5.1.* The idea of the proof is simple and we summarize it here (the precise details are given in §4.5.4 below). By the geometric interpretation of the monodromy around an isolated Lefschetz singularity - [Arn], see also [Sei2] - there exists a Lefschetz fibration $\pi : \mathcal{E} \rightarrow \mathbb{C}$ with a single singularity (chosen at the origin) and with general fiber X . Moreover, there is a cobordism $V \subset \mathcal{E}$ as in Figure 24, that projects to the curve γ'' there, and has ends Q and $\tau_S Q$. Consider a second cobordism W , as in the same picture, obtained as the trail of N along the curve γ' , where N is any Lagrangian in $\mathcal{L}^*(X)$. The cobordism techniques in [BC2] produce an associated chain morphism $CF(N, \tau_S Q) \rightarrow CF(N, Q)$ given by counting the Floer strips going from the intersections of W and V that project to w_1 to the intersections that project to w_0 and the cone - in the sense of chain complexes - over this morphism is $CF(W, V)$. The proof reduces to finding a quasi-isomorphism $CF(N, S) \otimes CF(S, Q) \rightarrow CF(W, V)$. The next step is again geometric and is based on the well-known fact that the function $\text{Re}(\pi)$ is Morse with a single singularity at the origin and that its gradient with respect to the standard metric is Hamiltonian. Moreover, the positive horizontal thimble originating at 0 is the stable manifold of $\text{Re}(\pi)$ and the negative horizontal thimble is the unstable manifold of $\text{Re}(\pi)$. To start this stage in the proof, we use the flow of $\nabla \text{Re}(\pi)$ to push W to the right in picture Figure 24 thus getting \widetilde{W} ; similarly, we use the gradient of $-\text{Re}(\pi)$ to push V to the left in the same picture thus getting \widetilde{V} - see Figure 25. We notice that $CF(\widetilde{W}, \widetilde{V}) \cong CF(W, V)$ and analyze the complex $CF(\widetilde{W}, \widetilde{V})$. Assuming all relevant intersections are generic, by standard Morse theory, if W is pushed enough to the right, \widetilde{W} intersects a neighborhood around the singularity in a number n_1 of copies of the stable manifold of $\text{Re}(\pi)$. Moreover, n_1 is equal to the number of intersections of W with the unstable manifold of $\text{Re}(\pi)$. Similarly, \widetilde{V} intersects the same neighborhood in n_2 copies of the unstable manifold of $\text{Re}(\pi)$ and n_2 is equal to the number of intersections of V with the stable manifold of $\text{Re}(\pi)$. The interpretation of the stable and unstable manifolds as thimbles (and our transversality assumptions) immediately imply that n_1 equals the number of intersection points $N \cap S$ and n_2 is the number of intersections $S \cap Q$. Moreover, each copy of the stable manifold that is associated to W' intersects precisely once each copy of the unstable manifold that is contained in V' . In short, it follows that there is a bijection Ξ between the following two sets $(N \cap S) \times (S \cap Q) \equiv (\widetilde{W} \cap \widetilde{V})$. The last step of the proof is more technical and shows that Ξ extends to a quasi-isomorphism of chain complexes. The basic idea here is to compare the bijection Ξ with the product $\mu_2 : CF(\widetilde{W}, T_\Delta) \otimes CF(T_\Delta, \widetilde{V}) \rightarrow CF(\widetilde{W}, \widetilde{V})$ where T_Δ is a thimble as in Figure 24. The key part of the argument is to notice that the J -holomorphic triangles giving this product decompose in two classes: “short” ones, of small area, and “long”

ones, of big area, and that the short component of μ_2 is a bijection identified to Ξ . Because we work over \mathcal{A} this means that the product μ_2 is a quasi isomorphism and the wanted statement easily follows.

Remark 4.5.4. The reason we restrict ourselves to $\dim_{\mathbb{R}} X \geq 4$ is the following. As mentioned above, the proof uses an auxiliary Lefschetz fibration \mathcal{E} with a single singularity and with general fiber X . Moreover, we will use a version of the Fukaya category of of cobordisms in \mathcal{E} . For this to work we need \mathcal{E} to be strongly monotone (see Definition 3.2.1). This easy follows from the monotonicity of X when $\dim_{\mathbb{R}} X \geq 4$. However, when $\dim_{\mathbb{R}} X = 2$ this might not be the case anymore. It seems plausible that this difficulty can be overcome (since in dimension 4 (i.e. the dimension of \mathcal{E}) for a generic almost complex structure there are no holomorphic disks with non-positive Maslov numbers.)

4.5.4. *Proof of Proposition 4.5.1.* The actual proof consists of seven steps that follow below. Two auxiliary Lemmas that are used along the way are proved in §4.5.5 and §4.5.6.

To fix ideas, we first carry out the proof under the assumption that X is closed. We discuss the non-compact case at the end.

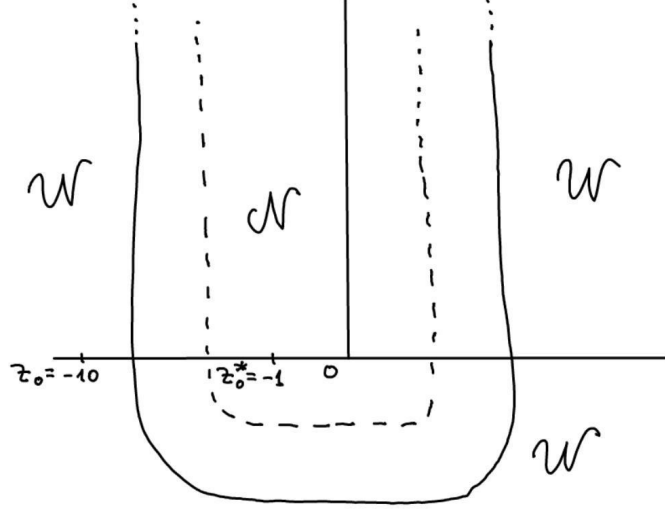
Step 1: *Constructing an appropriate Lefschetz fibration.*

We first claim that there exists a Lefschetz fibration $\pi : \mathcal{E} \rightarrow \mathbb{C}$ with symplectic structure Ω so that \mathcal{E} is tame over a subset $\mathcal{W} \subset \mathbb{C}$ as in Figure 23, and there is a symplectic trivialization ψ over \mathcal{W} (see Definition 2.2.2), such that \mathcal{E} , Ω and ψ have the following properties:

- (1) The fibration has only one critical point $p \in \mathcal{E}$ lying over $0 \in \mathbb{C}$.
- (2) The fiber $(\mathcal{E}_{z_0}, \Omega|_{\mathcal{E}_{z_0}})$ over $z_0 = -10 \in \mathbb{C}$ is symplectomorphic via the trivialization ψ to (X, ω) . (Henceforth we make this identification.)
- (3) The vanishing cycle in \mathcal{E}_{z_0} associated to the path going from z_0 to 0 along the x -axis is S .
- (4) The monodromy associated to a loop λ based at z_0 that goes around 0 counterclockwise is Hamiltonian isotopic to τ_S .

To prove this we first construct a Lefschetz fibration $\mathcal{E} \rightarrow \mathbb{C}$ (not necessarily tame) whose total space is endowed with a symplectic structure Ω^* and with the following properties:

- (1) The fibration has only one critical point $p \in \mathcal{E}$ lying over $0 \in \mathbb{C}$.
- (2) The fiber over $z_0^* = -1 \in \mathbb{C}$ is $(\mathcal{E}_{z_0^*}, \Omega|_{\mathcal{E}_{z_0^*}}) = (X, \omega)$.
- (3) The vanishing cycle in $\mathcal{E}_{z_0^*}$ associated to the path going from z_0^* to 0 along the x -axis is Hamiltonian isotopic to S .
- (4) The monodromy around a loop λ^* based at z_0^* which goes counterclockwise around the critical value 0 is Hamiltonian isotopic to the Dehn twist τ_S .

FIGURE 23. Constructing the fibration \mathcal{E} .

The proof that such a Lefschetz fibration exists follows from [Sei2] (see also Chapter 16e in [Sei3]), where it is proved for exact Lagrangian spheres. This is a local argument and therefore that proof extends to the case when X is possibly not exact.

Given the fibration $\mathcal{E} \rightarrow \mathbb{C}$ and Ω^* we apply Proposition 2.3.1 with appropriate subsets \mathcal{N} and \mathcal{W} as in Figure 23 and base point $z_0 = -10$. We obtain a new symplectic structure Ω' on \mathcal{E} with respect to which the fibration is tame over \mathcal{W} and such that Ω' coincides with Ω^* over \mathcal{N} . We thus obtain a trivialization $\psi' : (\mathcal{W} \times X', c\omega_{\mathbb{C}} \oplus \omega') \rightarrow (\mathcal{E}|_{\mathcal{W}}, \Omega')$, where $(X', \omega') = (\mathcal{E}|_{z_0}, \Omega^*|_{\mathcal{E}|_{z_0}})$ and $c > 0$.

Consider the loop λ which starts at z_0 , goes to z_0^* along the x -axis, then goes along λ^* and finally comes back to z_0 along the x -axis. Parallel transport along the straight segment connecting z_0 to z_0^* and with respect to the connection $\Gamma' = \Gamma(\Omega')$ gives a symplectomorphism $\varphi : (X', \omega') \rightarrow (X, \omega)$. Put $S' = \varphi^{-1}(S)$. Clearly the monodromy (with respect to Γ') along λ is $\varphi^{-1} \circ \tau_S \varphi = \tau_{S'}$.

Finally, the desired symplectic structure on \mathcal{E} and the trivialization are obtained by taking $\Omega = \Omega'$ and $\psi = \psi' \circ (\text{id} \times \varphi^{-1})$.

From now on the trivialization ψ will be implicitly assumed and we make the following identification

$$(\mathcal{E}|_{\mathcal{W}}, \Omega|_{\pi^{-1}(\mathcal{W})}) = (\mathcal{W} \times X, c\omega_{\mathbb{C}} \oplus \omega).$$

Step 2: *Translating the problem to cobordisms.*

First note that \mathcal{E} is strongly monotone of class $*$. This follows immediately from the Definition 3.2.1 (recall that we have assumed that $\dim_{\mathbb{R}} X \geq 4$) and Remark 3.2.4.

Let $\gamma' \subset \mathbb{C}$ be the curve depicted in Figure 24. In a similar way to [BC3] γ' gives rise to an inclusion functor

$$\mathcal{I}_{\gamma'} : \mathcal{Fuk}^*(X) \longrightarrow \mathcal{Fuk}^*(\mathcal{E})$$

whose action on objects is $\mathcal{I}_{\gamma'}(N) = \gamma'N$, where $\gamma'N \subset \mathcal{E}$ stands for the trail of N along the curve γ' (see §2.1.1). Here, by $\mathcal{Fuk}^*(\mathcal{E})$ we mean the Fukaya category of cobordisms in \mathcal{E} of monotonicity class $*$ but we do not require the cobordisms to be only negatively ended. This category is defined, following the recipe in [BC3] as described in §3.3, but by also using perturbations and bottlenecks associated to the positive ends. For the purpose of the proof below, it is actually enough to restrict to a subcategory whose objects are cobordisms in \mathcal{E} that project to curves in \mathbb{C} .

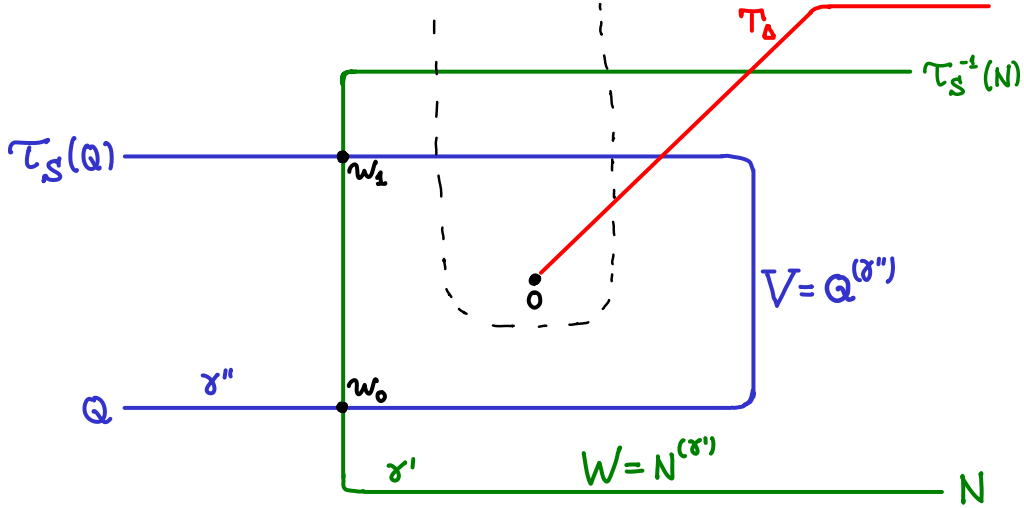


FIGURE 24. The cobordisms V , W and T_Δ .

Denote $W = \mathcal{I}_{\gamma'}N = \gamma'N$ and view it as a cobordism in \mathcal{E} . Next, consider the curve $\gamma'' \subset \mathbb{C}$ as depicted in Figure 24 and fix a base point $w_0 \in \gamma'' \cap \mathcal{W}$. Define $V \subset (\mathcal{E}, \Omega)$ to be the Lagrangian submanifold obtained as the trail of $Q \subset \mathcal{E}_{w_0} = X$ along γ'' . Clearly both V and W are monotone and by standard arguments (see [Che] and also [BC2, Remark 2.2.4]) we have $d_V = d_Q$ and $d_W = d_N$. It follows that both V and W are monotone of class $*$ hence are legitimate objects of the Fukaya category $\mathcal{Fuk}^*(\mathcal{E})$ as considered in this section.

Note that since the fibration (\mathcal{E}, Ω) is symplectically trivial over \mathcal{W} the lower end of V is identified with Q and due to the homotopy class of γ'' (in $(\mathbb{C} \setminus \{0\}, \text{rel } \infty)$) the upper end of V is a Lagrangian submanifold of X which is Hamiltonian isotopic to $\tau_S(Q)$. Similarly, the lower end of W is cylindrical over N and the upper end is cylindrical over $\tau_S^{-1}(N)$.

Below we will work with the Fukaya categories of both X and \mathcal{E} . Our choices of auxiliary parameters (Floer and perturbation data, etc.) for these categories will be as described in §3.

We therefore omit them from the notation in Floer complexes and the other A_∞ -structures. There are a few modifications compared to the conventions used in §3: we assume that the ends of the curve γ' are at height -2 and 2 and the ends of γ'' are at -1 and 1 . In other words, to fit precisely the setting in §3 we need to translate the whole picture by $+3i$. Clearly, this adjustment is formal and it has no impact on the construction of the relevant Fukaya categories (it is required because we prefer to keep the critical value of π to be at 0).

Denote by $\mathcal{Y}_X : \mathcal{Fuk}^*(X) \rightarrow \text{mod}(\mathcal{Fuk}^*(X))$ and $\mathcal{Y} : \mathcal{Fuk}^*(\mathcal{E}) \rightarrow \text{mod}(\mathcal{Fuk}^*(\mathcal{E}))$ the Yoneda embeddings associated to the Fukaya categories of X and \mathcal{E} respectively. When no confusion may arise we will simplify the notation and denote the module $\mathcal{Y}_X(L)$ associated to a Lagrangian $L \subset X$ simply by L and similarly for \mathcal{E} .

We now analyze the pullback module $\mathcal{I}_{\gamma'}^* V \in \text{mod}(\mathcal{Fuk}^*(X))$. Similar arguments to §4.4 [BC3] (see also §4.3 in this paper, in particular the exact sequence at Step 3 i.) show that we have a quasi-isomorphism:

$$(40) \quad \mathcal{I}_{\gamma'}^* V \simeq \text{cone}(\tau_S(Q) \xrightarrow{\varphi} Q),$$

for some homomorphism of A_∞ -modules φ that is induced by counting holomorphic strips (and polygons) going from the intersection of V with W at the $\tau_S(Q)$ end to the intersection of V and W at the Q end - see Figure 24.

Let $T_\Delta \subset \mathcal{E}$ be the thimble corresponding to the “diagonal” curve Δ depicted in Figure 24. By Proposition 3.2.3 T_Δ is monotone of class $(*)$ and we view it as an object of $\mathcal{Fuk}^*(\mathcal{E})$.

Consider now the $\mathcal{Fuk}^*(\mathcal{E})$ -module

$$(41) \quad \mathcal{M} = T_\Delta \otimes CF(T_\Delta, V),$$

where the second factor in the tensor product is regarded as a chain complex (see Chapter 3c in [Sei3] for the definition of the tensor product of an A_∞ -module and a chain complex).

The A_∞ -operations μ_k , $k \geq 2$, induce a homomorphism of modules $\mathcal{M} \rightarrow V$. Pulling back by $\mathcal{I}_{\gamma'}$, this homomorphism induces a homomorphism of $\mathcal{Fuk}^*(X)$ -modules:

$$(42) \quad \nu : \mathcal{I}_{\gamma'}^* \mathcal{M} \rightarrow \mathcal{I}_{\gamma'}^* V.$$

We claim that Proposition 4.5.1 reduces to the next statement:

Proposition 4.5.5. *The homomorphism ν is a quasi-isomorphism.*

This is due to the following quasi-isomorphisms:

$$(43) \quad \mathcal{I}_{\gamma'}^* \mathcal{M} = \mathcal{I}_{\gamma'}^* T_\Delta \otimes CF(T_\Delta, V) \simeq S \otimes CF(S, Q).$$

Here we identify S and its image under the Yoneda embedding.

In turn, by the general theory of A_∞ -categories, in order to prove Proposition 4.5.5 it is enough to show that for every Lagrangian $N \in \mathcal{Ob}(\mathcal{Fuk}^*(X))$ the map

$$(44) \quad \mu_2 : CF(\gamma'N, T_\Delta) \otimes CF(T_\Delta, V) \longrightarrow CF(\gamma'N, V)$$

is a quasi-isomorphism. (Recall that $\gamma'N$ stands for the trail of N along γ' .)

Remark 4.5.6. We have not indicated at this moment the choices of Floer and perturbation data in (44) for two reasons. This is because, whether or not the map in (44) is a quasi-isomorphism does not depend on these specific choices (the induced product in homology is canonical). Moreover, later on in the proof we will actually make use of a very specific choice of parameters (which is different than the one used in §3 when setting up the entire Fukaya category of \mathcal{E} !) for which it will be convenient to prove that the map in (44) is a quasi-isomorphism.

The rest of this section will be devoted to proving that (44) is a quasi-isomorphism. For brevity we denote from now on $W = \gamma'N \subset \mathcal{E}$ (see Figure 24).

Step 3: *Stretching the cobordisms.*

Write the projection $\pi : \mathcal{E} \longrightarrow \mathbb{C}$ as

$$\pi = \operatorname{Re}(\pi) + \operatorname{Im}(\pi)i.$$

Denote by $Z = -\nabla \operatorname{Re}(\pi)$ the negative gradient vector field of the real part of π with respect to the Riemannian metric induced on \mathcal{E} by $(\Omega, J_\mathcal{E})$. Since the functions $\operatorname{Re}(\pi)$ and $\operatorname{Im}(\pi)$ are harmonic conjugate (recall that π is holomorphic), it follows that Z is also the Hamiltonian vector field associated to the function $\operatorname{Im}(\pi)$.

The flow of the vector field Z will be used extensively throughout the rest of the proof. However, due to the non-compactness of \mathcal{E} , it might lack to be defined for all times. To overcome this difficulty we proceed as follows. Write $y_1 + iy_2 \in \mathbb{C}$ for the standard coordinates on \mathbb{C} . Denote by R^Ω the curvature of the connection $\Gamma(\Omega)$. (Recall that this is a 2-form on \mathbb{C} with values in the space of Hamiltonian functions of the fibers of \mathcal{E} .) A straightforward calculation shows that for every $z \in \mathbb{C}$, $p \in \mathcal{E}_z$ we have:

$$(45) \quad Z_{(z,p)} = \frac{-1}{C(z) - R_z^\Gamma(\partial_{y_1}, \partial_{y_2})(p)} (\partial_{y_1})^{\text{hor}},$$

where $C : \mathbb{C} \longrightarrow \mathbb{R}$ is a function and $(\partial_{y_1})^{\text{hor}}$ stands for the horizontal lift of ∂_{y_1} . Since $Z = -\nabla \operatorname{Re}(\pi)$ it follows that the denominator on the right-hand side of (45) is always positive. Fix a positive real number $a > 0$ and define

$$\Omega' = \Omega + a\pi^* dy_1 \wedge dy_2.$$

Note that $J_\mathcal{E}$ continues to be compatible with Ω' . Denote by Z' the negative gradient of the same function, $\operatorname{Re}(\pi)$, but now defined via the metric associated to $(\Omega', J_\mathcal{E})$. A simple

calculation shows that:

$$(46) \quad Z'_{(z,p)} = \frac{-1}{a + C(z) - R_z^\Gamma(\partial_{y_1}, \partial_{y_2})(p)} (\partial_{y_1})^{\text{hor}}.$$

Clearly the coefficient standing before $(\partial_{y_1})^{\text{hor}}$ on the right-hand side of (46) is bounded from above by $1/a$. It now easily follows that the flow of Z' exists for all times (recall that we are under the assumption that the fiber X is compact). Finally, note that the connections of Ω and Ω' are the same and moreover, V and W continue both to be Lagrangian cobordisms with respect to the new form Ω' .

Summarizing the preceding procedure, by replacing Ω by Ω' we may assume that the negative gradient flow of $\text{Re}(\pi)$ exists for all times. For simplicity we continue to denote the symplectic structure of \mathcal{E} by Ω .

Denote by ϕ_t , $t \in \mathbb{R}$, the flow of Z . Note that the function $\text{Re}(\pi)$ is a Morse function with exactly one critical point $p \in \mathcal{E}$ lying over $0 \in \mathbb{C}$. The Morse index of $\text{Re}(\pi)$ at p is precisely $n + 1 = \dim_{\mathbb{C}} \mathcal{E}$. Denote by ϕ_t , $t \in \mathbb{R}$, the flow of Z . The stable submanifold of Z is the thimble T' lying over the positive x -axis and the unstable submanifold of Z is the thimble T'' lying over the negative x -axis. Note that we have $J_{\mathcal{E}} T_p(T') = T_p(T'')$.

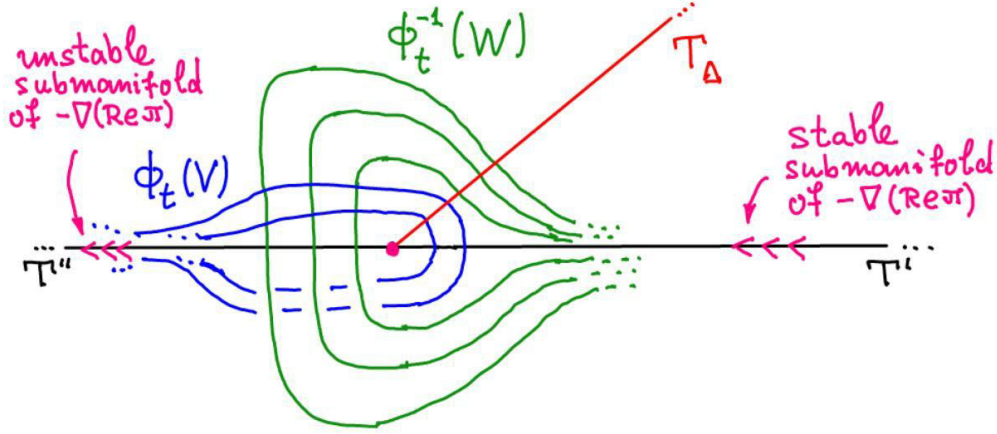


FIGURE 25. The cobordisms V , W after the flows ϕ_t and ϕ_t^{-1} are applied to them for large time t .

Denote by $B'(r) = B''(r) = B^{n+2}(r) \subset \mathbb{R}^{n+2}$ two copies of the $n + 1$ -dimensional closed Euclidean ball of radius r around $0 \in \mathbb{R}^{n+2}$. (Since each of these two balls corresponds to a different factor of $\mathbb{R}^{n+2} \times \mathbb{R}^{n+2}$ we have chosen to denote them by different symbols.)

Fix a little neighborhood $Q_p \subset \mathcal{E}$ of p which is symplectomorphic to a product $B'(r_0) \times B''(r_0) \subset (\mathbb{R}^{n+2} \times \mathbb{R}^{n+2}, \omega_{\text{can}} = dp_1 \wedge dq_1 + \cdots dp_m \wedge dq_m)$ for some small r_0 . Below we will abbreviate $B' = B'(r_0)$, $B'' = B''(r_0)$.

We may assume that the symplectic identification $Q_p \approx B' \times B''$ sends $T' \cap Q_p$ to $B' \times \{0\}$ and $T'' \cap Q_p$ to $\{0\} \times B''$ and T_Δ to the diagonal $\{(x, y) \in B' \times B'' \mid x = y\}$. From now on we assume the identification $Q_p \approx B' \times B''$ explicit and when convenient view Q_p as a subset of \mathbb{R}^{2m} .

We now apply the flow ϕ_t to V and ϕ_t^{-1} to W (see Figures 25, 26). Standard arguments in Morse theory imply that for $t_0 \gg 1$ we have

$$\phi_{t_0}^{-1}(W) \cap Q_p = \coprod_{i=1}^{s''} D'_i, \quad \phi_{t_0}(V) \cap Q_p = \coprod_{j=1}^{s'} D''_j,$$

where $D'_i \subset Q_p$ are graphs of exact 1-forms on B' and $D''_j \subset Q_p$ are graphs of exact 1-forms on B'' . Here $s'' = \#(W \cap T'')$ and $s' = \#(V \cap T')$ are the number of intersection points (counted without signs) of $W \cap T''$ and $V \cap T'$ respectively. Note also that by our construction of \mathcal{E} we have $s'' = \#(N \cap S)$ and $s' = \#(Q \cap S')$, where S' is the vanishing sphere $T' \cap \mathcal{E}_x$ with $0 < x$ large enough so that $x \in \mathcal{W}$. Note that S' , when viewed as a Lagrangian in (X, ω) is Hamiltonian isotopic to S .

Fix $0 < \delta_0 \ll 1/3$. By taking t_0 large enough we may assume that

$$(47) \quad \phi_{t_0}^{-1}(W) \cap Q_p \subset B' \times B''(\delta_0 r_0), \quad \phi_{t_0}(V) \cap Q_p \subset B'(\delta_0 r_0) \times B''$$

and moreover that each of the D'_i (resp. D''_j 's) is C^1 -close to a constant section of $B' \times B'' \rightarrow B'$ (resp. $B' \times B'' \rightarrow B''$). See Figure 26.

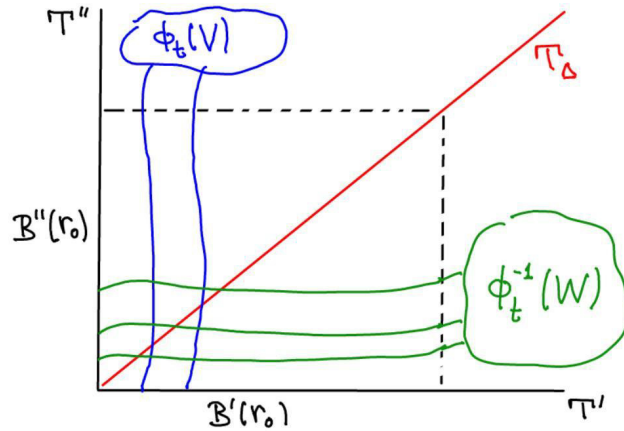


FIGURE 26. The parts of $\phi_t(V)$ and $\phi_t^{-1}(W)$ that lie in Q_p .

Thus by applying a suitable Hamiltonian diffeomorphism of Q_p (which extends to the rest of \mathcal{E}) we may assume that

$$\phi_{t_0}^{-1}(W) \cap Q_p = \coprod_{i=1}^{s''} B' \times \{a_i''(t_0)\}, \quad \phi_{t_0}(V) \cap Q_p = \coprod_{j=1}^{s'} \{a_j'(t_0)\} \times B'',$$

where $|a_i'(t_0)|, |a_j''(t_0)| < \delta_0 r_0$. See Figure 27.

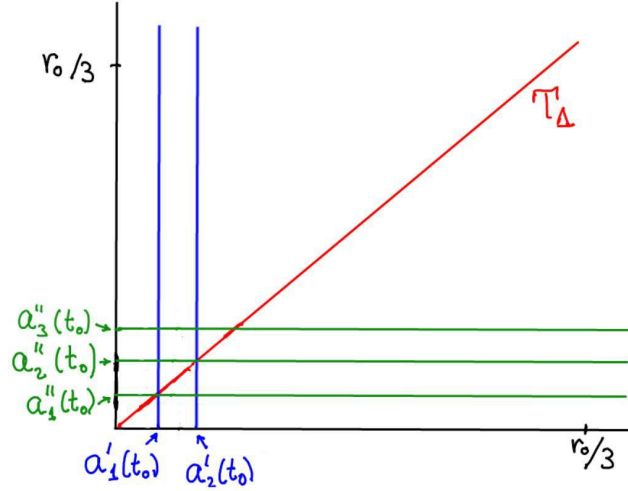


FIGURE 27. Isotoping $\phi_{t_0}(V) \cap Q_p$ and $\phi_{t_0}^{-1}(W) \cap Q_p$ to be constant sections.

Fix now t_0 large enough as above and set

$$\tilde{V} := \phi_{t_0}(V), \quad \tilde{W} = \phi^{-1}(t_0)(W).$$

For $r', r'' < r_0$ we abbreviate $B(r', r'') := B'(r') \times B''(r'')$ and also $B = B(r_0, r_0) = B' \times B''$.

Step 4: *A further isotopy of \tilde{V} and \tilde{W} .*

We claim there exist two Hamiltonian isotopies ψ_t', ψ_t'' , $0 \leq t < 1$, with $\psi_0' = \psi_0'' = \text{id}$ and with the following properties for every $0 \leq t < 1$:

- (1) ψ_t', ψ_t'' are both supported in $\text{Int}(B)$.
- (2) $\psi_t'(\tilde{W}) \cap B(r_0/3, r_0/3) = \coprod_{i=1}^{s''} B'(r_0/3) \times \{b_i''(t)\}$ with $|b_i''(t)| \leq (1-t)\delta_0 r_0$ for every i .
- (3) $\psi_t''(\tilde{V}) \cap B(r_0/3, r_0/3) = \coprod_{j=1}^{s'} \{b_j'(t)\} \times B''(r_0/3)$ with $|b_j'(t)| \leq (1-t)\delta_0 r_0$ for every j .
- (4) $\psi_t'(\tilde{W}) \cap \left((B'(r_0) \setminus B'(2r_0/3)) \times B''(r_0) \right) = \tilde{W} \cap \left((B'(r_0) \setminus B'(2r_0/3)) \times B''(r_0) \right)$.
- (5) $\psi_t''(\tilde{V}) \cap \left(B'(r_0) \times (B''(r_0) \setminus B''(2r_0/3)) \right) = \tilde{V} \cap \left(B'(r_0) \times (B''(r_0) \setminus B''(2r_0/3)) \right)$.
- (6) $\psi_t'(\tilde{W})$ and $\psi_t''(\tilde{V})$ intersect only inside $B(\delta_0 r_0, \delta_0 r_0)$. Moreover, their intersection is: $\psi_t'(\tilde{W}) \cap \psi_t''(\tilde{V}) = \{(b_j'(t), b_i''(t)) \mid 1 \leq i \leq s'', 1 \leq j \leq s'\}$.

$$(7) \quad T_\Delta \cap \psi_t''(\widetilde{V}) \subset B(r_0/3, r_0/3) \text{ and } T_\Delta \cap \psi_t'(\widetilde{W}) \subset B(r_0/3, r_0/3). \quad .$$

See Figure 28. The construction of the isotopies ψ_t', ψ_t'' is elementary and can be done quite explicitly. For point (7) one might need to reduce the size of the parameter δ_0 from (47), which can be done in advance.

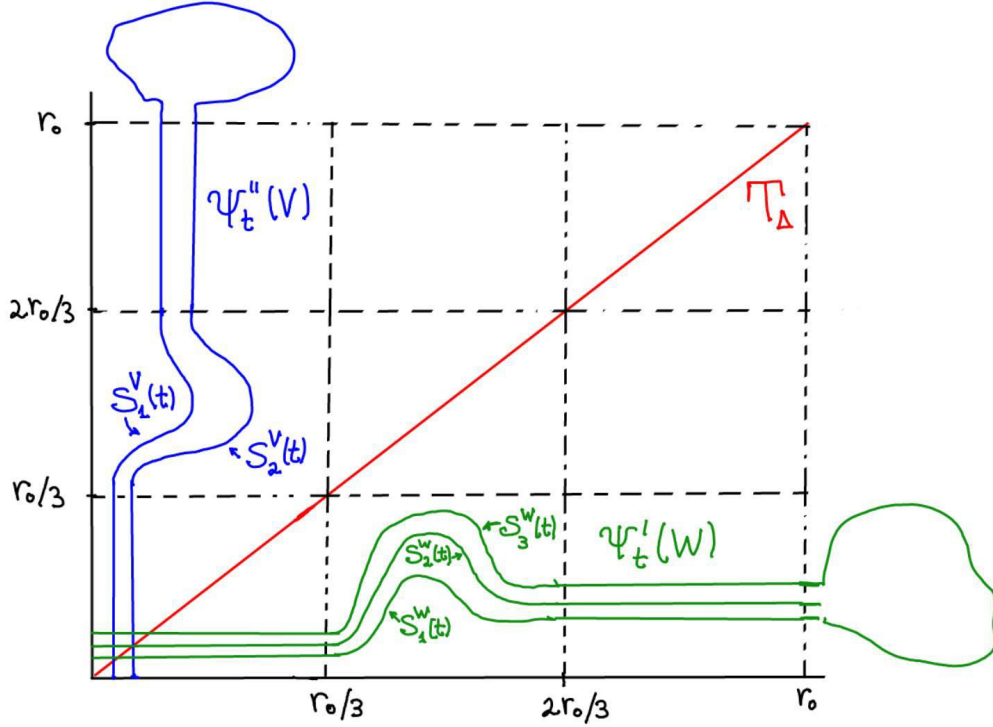


FIGURE 28. The isotopies $\psi_t'(\widetilde{V}), \psi_t''(\widetilde{W})$

To keep the notation short we now set:

$$\widetilde{V}_t = \psi_t''(\widetilde{V}), \quad \widetilde{W}_t = \psi_t'(\widetilde{W}).$$

Note that $\widetilde{W}_t \cap B(r_0, r_0)$ is *disconnected* and has precisely s'' connected components, each of which looks like a copy of $B' \times \{0\}$ which is (non-linearly) translated along the B'' -axis. These components lie in “parallel” position one with respect to the other (see Figure 28). We will refer to these components as the *sheets* of \widetilde{W}_t inside $B(r_0, r_0)$ and denote them by $\mathcal{S}_i^W(t)$, $i = 1, \dots, s''$. The indexing here is so that $\mathcal{S}_i^W(t)$ coincides with $B'(r_0/3) \times \{b_i''(t)\}$ inside $B(r_0/3, r_0/3)$. Similarly, $\widetilde{V}_t \cap B(r_0, r_0)$ is *disconnected* and consists of s' “parallel” sheets which are all “translates” of $\{0\} \times B''$. We denote them by $\mathcal{S}_j^V(t)$, $j = 1, \dots, s'$, where the indexing is done so that $\mathcal{S}_j^V(t)$ coincides with $\{b_j(t)\} \times B''(r_0/3)$ inside $B'(r_0/3, r_0/3)$. See

Figure 28. Clearly we have

$$(48) \quad \begin{aligned} \mathcal{S}_i^W(t) \cap \mathcal{S}_j^V(t) &= \{(b'_j(t), b''_i(t))\}, \\ \mathcal{S}_i^W(t) \cap T_\Delta &= \{(b''_i(t), b''_i(t))\}, \quad T_\Delta \cap \mathcal{S}_j^V(t) = \{(b'_j(t), b'_j(t))\}. \end{aligned}$$

Step 5: *Area estimates for large holomorphic triangles.* Let $D' = D \setminus \{z_1, z_2, z_3\}$ be the unit disk punctured at three boundary points z_1, z_2, z_3 ordered clock-wise along ∂D . Fix strip-like ends around the punctures (see §3), and denote by $\partial_{i,j}D'$, the arc on $\partial D'$ connecting z_i with z_j .

We will now consider a special almost complex structure J_B^0 on $B = B' \times B''$. We identify $\mathbb{R}^{n+2} \times \mathbb{R}^{n+2}$ with \mathbb{C}^{n+2} in the obvious way via $(x_1, \dots, x_m, y_1, \dots, y_{n+2}) \mapsto (x_1 + iy_1, \dots, x_{n+2} + iy_{n+2})$. This induces a complex structure J_{std} on $\mathbb{R}^{n+2} \times \mathbb{R}^{n+2}$. We define J_B^0 to be the restriction of J_{std} to $B \subset \mathbb{R}^{n+2} \times \mathbb{R}^{n+2}$. Define now \mathcal{J}_0 to be the space of Ω -compatible domain-dependent almost complex structures $J = \{J_z\}_{z \in D'}$ which coincide with J_B^0 on B . For elements $J \in \mathcal{J}_0$, and $z \in D'$, $p \in \mathcal{E}$ we will also write $J(z, p)$ for the restriction of J_z to $T_p\mathcal{E}$.

Consider now finite energy solutions to the Floer equation with boundary conditions on the Lagrangians $\widetilde{W}_t, T_\Delta, \widetilde{V}_t$:

$$(49) \quad \begin{aligned} u : D' &\longrightarrow \mathcal{E}, \quad E(u) < \infty, \\ Du + J(z, u) \circ Du \circ j &= 0, \\ u(\partial_{3,1}D') &\subset \widetilde{W}_t, \quad u(\partial_{1,2}D') \subset T_\Delta, \quad u(\partial_{2,3}D') \subset \widetilde{V}_t \end{aligned}$$

together with the requirement that u converges along each strip-like end of D' to an intersection point between the corresponding pair of Lagrangians (associated to the two arcs of $\partial D'$ that neighbor a given puncture). Thus u extends continuously to a map $u : D \longrightarrow \mathcal{E}$ with

$$u(z_1) \in \widetilde{W}_t \cap T_\Delta, \quad u(z_2) \in T_\Delta \cap \widetilde{V}_t, \quad u(z_3) \in \widetilde{W}_t \cap \widetilde{V}_t.$$

In what follows we denote for a (finite energy) map $u : D \longrightarrow \mathcal{E}$ by $A_\Omega(u) = \int_{D'} u^*\Omega$ its symplectic area.

We now fix once and for all r_1 with $2r_0/3 < r_1 < r_0$.

Lemma 4.5.7. *There exists a constant $C = C(r_1, \widetilde{W}, \widetilde{V}) > 0$ (that depends only on r_1 and $\widetilde{W} = \widetilde{W}_0, \widetilde{V} = \widetilde{V}_0$) with the following property. Let $0 \leq t < 1$ and $J \in \mathcal{J}_0$. Then every solution $u : D' \longrightarrow \mathcal{E}$ of (49) with $u(D') \not\subset B(r_1, r_1)$ must satisfy $A_\Omega(u) \geq C$.*

The proof of the lemma is given in §4.5.5 below.

Next consider the intersections between any of $\widetilde{W}_t, \widetilde{V}_t$ and T_Δ . Recall from (48) the intersections between $\mathcal{S}_i^W(t), \mathcal{S}_j^V(t)$ and T_Δ . For simplicity we set

$$w_i(t) = (b''_i(t), b'_i(t)), \quad v_j(t) = (b'_j(t), b'_j(t)), \quad x_{i,j}(t) = (b'_j(t), b''_i(t)).$$

With this notation we have:

$$(50) \quad \begin{aligned} \widetilde{W}_t \cap T_\Delta &= \{w_i(t) \mid 1 \leq i \leq s''\}, & T_\Delta \cap \widetilde{V}_t &= \{v_j(t) \mid 1 \leq j \leq s'\}, \\ \widetilde{W}_t \cap \widetilde{V}_t &= \{x_{i,j}(t) \mid 1 \leq i \leq s'', 1 \leq j \leq s'\}. \end{aligned}$$

As a consequence from Lemma 4.5.7 we have:

Corollary 4.5.8. *Let $0 \leq t < 1$, $J \in \mathcal{J}_0$ and $u : D' \longrightarrow \mathcal{E}$ a solution of (49). If*

$$u(z_1) = w_i(t), \quad u(z_2) = v_j(t), \quad u(z_3) \neq x_{i,j}(t),$$

then $A_\Omega(u) \geq C$, where C is the constant from Lemma 4.5.7.

Proof of Corollary 4.5.8. Let $u : D' \longrightarrow \mathcal{E}$ be as in the statement of the corollary. We claim that $u(\partial D') \not\subset B(r_1, r_1)$.

To prove this, assume the contrary were the case, i.e. that $u(\partial D') \subset B(r_1, r_1)$. Since $u(z_1) = w_i(t)$ it follows that $u(\partial_{1,3} D') \subset \mathcal{S}_i^W(t)$. Similarly, from $u(z_2) = v_j(t)$ we conclude that $u(\partial_{1,2} D') \subset \mathcal{S}_j^V(t)$. It now follows that $u(z_3) \in \mathcal{S}_i^W(t) \cap \mathcal{S}_j^V(t) = \{x_{i,j}(t)\}$, which is a contradiction. This proves that $u(\partial D') \not\subset B(r_1, r_1)$. By Lemma 4.5.7 we have $A_\Omega(u) \geq C$. \square

Step 6: *Estimating the small holomorphic triangles.*

Lemma 4.5.9. *There exists $\epsilon > 0$ and a constant $C' > 0$ such that the following holds. Let $1 - \epsilon \leq t < 1$ and $1 \leq i \leq s''$, $1 \leq j \leq s'$ and $J \in \mathcal{J}_0$. Then among the solutions of equation (49) there exists a unique one u with the following two properties:*

- (1) $u(z_1) = w_i(t)$, $u(z_2) = v_j(t)$, $u(z_3) = x_{i,j}(t)$.
- (2) $A_\Omega(u) < C'$.

Moreover, this solution u satisfies $u(D') \subset B(r_0/3, r_0/3)$ and $A_\Omega(u) \leq \sigma(t)$, where $\sigma(t) \xrightarrow[t \rightarrow 1^-]{} 0$. Furthermore J is regular for the solution u in the sense that the linearized $\bar{\partial}$ operator is surjective at u .

The proof is given in §4.5.6 below.

Step 7: *End of the proof.* We are now ready to prove that the map in (44) is a quasi-isomorphism, thus proving Proposition 4.5.5.

Following Steps 1-6 above it is enough to show that the map

$$(51) \quad \mu_2 : CF(\widetilde{W}_t, T_\Delta) \otimes CF(T_\Delta, \widetilde{V}_t) \longrightarrow CF(\widetilde{W}_t, \widetilde{V}_t)$$

is a quasi-isomorphism for some $0 \leq t < 1$.

Next, note that the whether or not (51) (or (44)) is a quasi-isomorphism is independent of the Floer and perturbation data used for the respective Floer complexes and for the operation μ_2 . Therefore for the sake of our proof any choice of such data would do as long as it is regular and amenable to the situation of cobordisms. (In contrast, consistency with respect to the

perturbation data used for the higher μ_k 's is irrelevant for our present purposes.) We therefore choose for (51) Floer data for which the Hamiltonian perturbations are 0 and $J \in \mathcal{J}_0$.

By construction, $CF(\widetilde{W}_t, T_\Delta)$ has the elements $w_1(t), \dots, w_{s''}(t)$ as a basis. Similarly $CF(T_\Delta, \widetilde{V}_t)$ has a basis consisting of $v_1(t), \dots, v_{s'}(t)$ and $CF(\widetilde{W}_t, \widetilde{V}_t)$ can be endowed with the basis $\{x_{i,j}(t)\}_{1 \leq i \leq s'', 1 \leq j \leq s'}$. Thus we have a 1-1 correspondence between the associated basis of $CF(\widetilde{W}_t, T_\Delta) \otimes CF(T_\Delta, \widetilde{V}_t)$ and the basis of $CF(\widetilde{W}_t, \widetilde{V}_t)$, given by

$$w_i(t) \otimes v_j(t) \longmapsto x_{ij}(t).$$

We will now show that for $t < 1$ close enough to 1 and appropriate J , the matrix of μ_2 with respect to these bases is invertible. This will prove that for such a choice of t and J , μ_2 is in fact a chain isomorphism (hence a quasi-isomorphism for any other choice). Below we will denote the matrix of μ_2 with respect to these bases by M .

Fix a generic $J \in \mathcal{J}_0$ and t_0 with $1 \leq t_0 < 1 - \epsilon$ such that $\sigma(t_0) \ll C'$, where ϵ , C' and σ are as in Proposition 4.5.9. By Proposition 4.5.9 the entries in the diagonal of M have the form

$$M_{k,k}(T) = T^{\alpha_k} + O(T^{C'}),$$

with $0 \leq \alpha_k \leq \sigma(t_0)$. Here $o(T^{C'})$ stands for an element of the Novikov ring in which every monomial is of the form $c_l T^{\lambda_l}$ with $c_l \in \mathbb{Z}_2$ and $\lambda_l \geq C'$.

Similarly, by Corollary 4.5.8, the elements of M that are off the diagonal are all of the form

$$M_{k,l} = O(T^C), \quad \forall k \neq l,$$

where C is the constant from Corollary 4.5.8 and Lemma 4.5.7.

By choosing t_0 close enough to 1 we obtain α_k as close as we want to 0. It easily follows that for such a choice of t_0 the matrix M can be transformed via elementary row operations to an upper triangular matrix with non-zero elements in the diagonal. It follows that M is invertible. \square

Remark 4.5.10. It is not difficult to see that the map φ from (40) is chain-homotopic to the corresponding map constructed by Seidel (in the exact case) in his construction of the exact triangle associated to a Dehn twist. As a consequence, the exact triangle constructed above coincides with his.

4.5.5. *Proof of Lemma 4.5.7.* Throughout the proof we will denote by $\text{Ball}_x(r) \subset \mathbb{R}^{n+2} \times \mathbb{R}^{n+2}$ the open Euclidean ball of radius r centered at x .

Fix r_2 with $2r_0/3 < r_2 < r_1$ and let $\rho_2 > 0$ small enough so that:

- (1) For i and every $x \in \mathcal{S}_i^W(t) \cap (\partial B'(r_2) \times B'')$ the closed ball $\overline{\text{Ball}_x(\rho_1)}$ is disjoint from all $\mathcal{S}_k^W(t)$ for every $k \neq i$ as well as from \widetilde{W}_t and from T_Δ .
- (2) For j and every $x \in \mathcal{S}_j^V(t) \cap (B' \times \partial B''(r_2))$ the closed ball $\overline{\text{Ball}_x(\rho_1)}$ is disjoint from all $\mathcal{S}_k^V(t)$ for every $k \neq j$ as well as from \widetilde{V}_t and from T_Δ .

- (3) For every $x \in T_\Delta \cap (\partial B'(r_2) \times \partial B''(r_2))$ the closed ball $\overline{\text{Ball}_x(\rho_1)}$ is disjoint from \widetilde{W}_t and \widetilde{V}_t .

By construction, such a ρ_1 exists and can be chosen to be *independent* of $0 \leq t < 1$. (Recall that $\widetilde{W}_t \cap (B(r_0, r_0) \setminus B(2r_0/3, r_0))$ is independent of t .) See Figure 29.

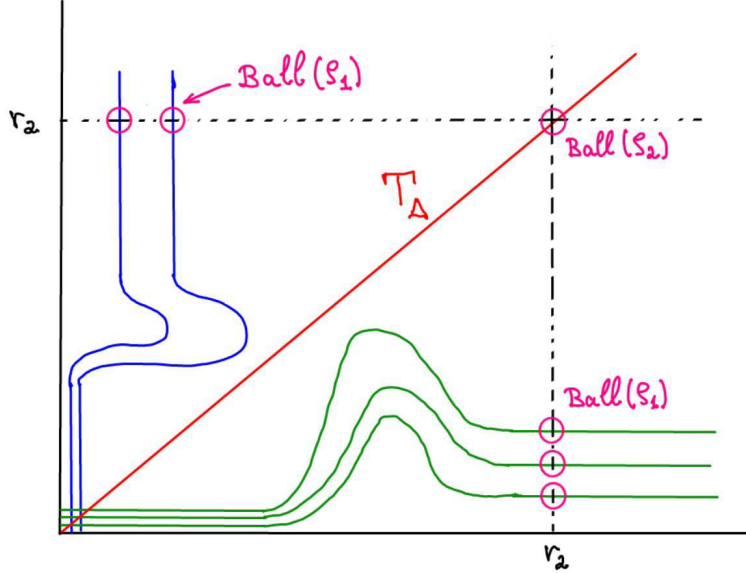


FIGURE 29. Estimating the area of holomorphic curves that go out of $B(2r_0/3, 2r_0/3)$.

Similarly, choose $\rho_2 > 0$ such that for every $x \in \partial B(r_1, r_1)$ the closed ball $\overline{\text{Ball}_x(\rho_2)}$ is disjoint from $B(r_2, r_2)$ and is also contained inside $B = B(r_0, r_0)$.

Set $C := \min\{\frac{\pi}{2}\rho_2^2, \pi\rho_1^2\}$.

Now let $u : D' \rightarrow \mathcal{E}$ be a solution of (49) and assume first that u satisfies the following special assumption: $u(\partial D') \not\subset B(r_2, r_2)$. We will prove that $A_\Omega(u) \geq C$.

Since $u(z_i) \in B(2r_0/3, 2r_0/3)$ (recall z_i are the punctures of D') it follows that there exists $z_* \in \partial D'$ such that $u(z_*)$ lies in one of the following three:

- (1) $\mathcal{S}_i^W(t) \cap (\partial B'(r_2) \times B'')$ for some i ; or
- (2) $\mathcal{S}_j^V(t) \cap (B' \times \partial B''(r_2))$ for some j ; or
- (3) $T_\Delta \cap (\partial B'(r_2) \times \partial B''(r_2))$.

Consider now the intersection $u(D') \cap \overline{\text{Ball}_{u(z_*)}(\rho_2)}$. By the Lelong inequality (applied after a reflection in the ball with respect to the corresponding Lagrangian) it follows that

$$A_\Omega(u) \geq \frac{\pi}{2}\rho_2^2 \geq C.$$

(Alternatively one can use an appropriate version of the monotonicity lemma for minimal surfaces to obtain the same inequality.) We have thus proved the lemma under the assumption that $u(\partial D') \not\subset B(r_2, r_2)$.

We are now ready to prove the general case. Assume that $u(D') \not\subset B(r_1, r_1)$. There are two cases (mutually not exclusive): either $u(\partial D') \not\subset B(r_1, r_1)$, or $u(\text{Int } D') \not\subset B(r_1, r_1)$.

If the first case occurs then clearly $u(\partial D') \not\subset B(r_2, r_2)$ and we are done. Therefore we may assume that $u(\partial D') \subset B(r_2, r_2)$ and that the second case occurs, namely $u(\text{Int } D') \not\subset B(r_1, r_1)$. It follows that there is $z_* \in \text{Int } D'$ with $u(z_*) \in \partial B(r_1, r_1)$. Applying the Lelong inequality for $u(D') \cap \overline{\text{Ball}_{u(z_*)}(\rho_1)}$ we obtain

$$A_\Omega(u) \geq \pi \rho_1^2 \geq C.$$

□

4.5.6. Proof of Lemma 4.5.9. Before defining the constant C' , we first consider solutions u of (49) that satisfy property (1) of our proposition as well as property (2) with the constant C' replaced by the constant C from Lemma 4.5.7. (The constant C' , defined below, will have the property that $0 < C' \leq C$.) By Lemma 4.5.7 we have $u(D') \subset B(r_1, r_1)$. Since

$$\widetilde{W}_t \cap B(r_1, r_2) = \prod_{k=1}^{s''} \mathcal{S}_k^W(t), \quad \widetilde{V}_t \cap B(r_1, r_2) = \prod_{k=1}^{s'} \mathcal{S}_k^V(t)$$

it follows that

$$(52) \quad u(\partial_{3,1} D') \subset \mathcal{S}_i^W(t), \quad u(\partial_{1,2} D') \subset T_\Delta, \quad u(\partial_{2,3} D') \subset \mathcal{S}_j^V(t).$$

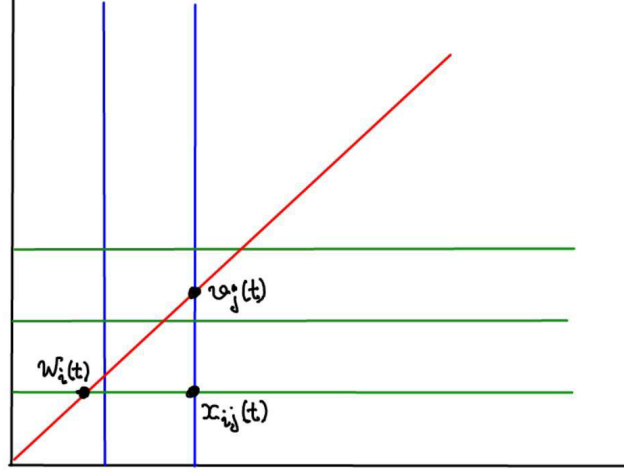
Thus we are considering here finite energy solutions $u : D' \rightarrow B' \times B''$ of (49) subject to the boundary condition (52) and the asymptotics (see Figure 30

$$(53) \quad u(z_1) = w_i(t), \quad u(z_2) = v_j(t), \quad u(z_3) = x_{i,j}(t).$$

Recall also that our almost complex structure J is in \mathcal{J}_0 , hence by definition $J \equiv J_B^0$ on $B = B' \times B''$.

We now claim that there is a constant $0 < C' \leq C$ such that all solutions u of (52) with asymptotics (53) and with $A_\Omega(u) \leq C'$ must satisfy $u(D') \subset B(r_0/3, r_0/3)$. The proof of this claim is very similar to that of Lemma 4.5.7 and in fact even simpler since we are considering here boundary conditions only on one pair of sheets $(\mathcal{S}_i^W(t), \mathcal{S}_j^V(t))$ and T_Δ , and the distance between each pair of these three Lagrangians outside of $B(r_0/3, r_0/3)$ is uniformly bounded below.

This proves that all solutions $u : D' \rightarrow \mathcal{E}$ that satisfy assumptions (1) and (2) of our proposition have their images inside $B(r_0/3, r_0/3)$.

FIGURE 30. Holomorphic triangles going from $w_i(t), v_j(t)$ to $x_{i,j}(t)$.

It remains to show the existence and uniqueness of such solutions, the area estimate and the regularity. To this end, set:

$$\mathcal{S}^W(t) := \mathbb{R}^m \times \{b''_i(t)\}, \quad \mathcal{S}^V(t) := \{b'_j(t)\} \times \mathbb{R}^m, \quad \mathcal{T}_\Delta = \{(x, x) \mid x \in \mathbb{R}^m\}.$$

Clearly $\mathcal{S}^W_i(t)$ coincides with $\mathcal{S}^W(t)$ inside $B(r_0/3, r_0/3)$ and similarly for $\mathcal{S}^V_j(t)$ and $\mathcal{S}^V(t)$ as well as for T_Δ and \mathcal{T}_Δ . Thus for our purposes we can consider now the equation (49) for maps $u : D' \longrightarrow \mathbb{R}^m \times \mathbb{R}^m$ with $J = J_{\text{std}}$ and with the following boundary condition and asymptotics:

$$(54) \quad \begin{aligned} u(\partial_{3,1}D') &\subset \mathcal{S}^W(t), \quad u(\partial_{1,2}D') \subset \mathcal{T}_\Delta(t), \quad u(\partial_{2,3}D') \subset \mathcal{S}^V(t), \\ u(z_1) &= w_i(t), \quad u(z_2) = v_j(t), \quad u(z_3) = x_{i,j}(t). \end{aligned}$$

Note that this problem splits. If we rearrange the coordinates by identifying of $\mathbb{R}^m \times \mathbb{R}^m \cong (\mathbb{R}^2)^{\times m}$ via the symplectic isomorphism $(p_1, \dots, p_m, q_1, \dots, q_m) \longmapsto (p_1, q_1, \dots, p_m, q_m)$ then J_{std} is sent to the standard split complex structure (which we continue to denote J_{std}), and $\mathcal{S}^W(t)$ becomes $(\mathbb{R} \times q_1(t)) \times \dots \times (\mathbb{R} \times q_m(t))$, where $b''_i(t) = (q_1(t), \dots, q_m(t))$. Similarly $\mathcal{S}^V(t)$ becomes $(p_1(t) \times \mathbb{R}) \times \dots \times (p_m(t) \times \mathbb{R})$, where $b'_j(t) = (p_1(t), \dots, p_m(t))$. Finally, \mathcal{T}_Δ becomes $\Delta_1 \times \dots \times \Delta_m$ where Δ_i is the diagonal in each of the \mathbb{R}^2 factors. We continue to denote the corresponding three Lagrangians by $\mathcal{S}^W(t)$, $\mathcal{S}^V(t)$ and \mathcal{T}_Δ .

We will now write maps $u : D' \longrightarrow (\mathbb{R}^2)^{\times m}$ as: $u(z) = (u_1(z), \dots, u_m(z))$ with $u_k(z) \in \mathbb{R}^2$. Clearly each of the maps $u_k : D' \longrightarrow \mathbb{R}^2 \cong \mathbb{C}$ is holomorphic (in the usual sense) and satisfies the boundary conditions and asymptotics (see Figure 31):

$$(55) \quad \begin{aligned} u(\partial_{3,1}D') &\subset \mathbb{R} \times q_k(t), \quad u(\partial_{1,2}D') \subset \Delta_k, \quad u(\partial_{2,3}D') \subset p_k(t) \times \mathbb{R}, \\ u(z_1) &= (q_k(t), q_k(t)), \quad u(z_2) = (p_k(t), p_k(t)), \quad u(z_3) = (p_k(t), q_k(t)). \end{aligned}$$

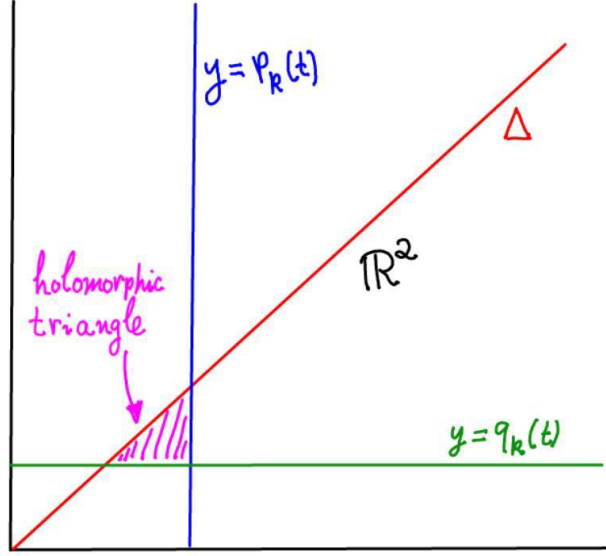


FIGURE 31. Holomorphic triangles in \mathbb{R}^2 corresponding to the projection on the k 'th factor of u .

Standard 1-dimensional complex analysis show that there is a unique holomorphic map $u_k^0 : D' \rightarrow \mathbb{C} \cong \mathbb{R}^2$ with the boundary conditions (55), the image of which is precisely the triangle consisting of the convex hull of the three points $(q_k(t), q_k(t))$, $(p_k(t)p_k(t))$, $(p_k(t), q_k(t))$. Moreover, a straightforward calculation (using e.g. the methods from Chapter 13 of [Sei3]) shows that the Maslov index of u_k^0 is 0 and that the standard complex structure of \mathbb{C} is regular for this solution.

Note that the mutual position of the three Lagrangians from (55) plays a crucial role here. If for example, one would replace Δ_k by the anti-diagonal line $\{(x, -x) : x \in \mathbb{R}\}$ then there would be no solutions with the boundary conditions (55), the reason being that the order of the punctures z_1, z_2, z_3 on ∂D is “wrong”.)

It follows that $u^0(z) = (u_1^0(z), \dots, u_m^0(z))$ is the unique holomorphic map $u : D' \rightarrow (\mathbb{R}^2)^{\times m}$ satisfying (54). Since the $\bar{\partial}$ -operator splits in a compatible way with the splitting $(\mathbb{R}^2)^{\times m}$ it follows that the index of u^0 is 0 and that J_{std} is regular.

Finally, it is clear that the symplectic area $A_\Omega(u^0)$ of u^0 is the sum of the areas of the triangles u_k^0 , $k = 1, \dots, m$. Since $p_k(t), q_k(t) \xrightarrow[t \rightarrow 1^-]{} 0$ it follows that $A_\Omega(u^0) \xrightarrow[t \rightarrow 1^-]{} 0$.

This concludes the proof of the proposition.

Remark. An alternative calculation of the index and regularity can be done by degenerating the problem to $t = 1$. Then the three Lagrangians forming the boundary conditions in (55) become $\mathbb{R} \times \{0\}$, Δ_k and $\{0\} \times \mathbb{R}$. The asymptotics at the punctures become $u_k(z_1) = u_k(z_2) = u_k(z_3) = (0, 0)$. It is easy to see that the only solution now is the constant solution at $(0, 0)$. The fact that its index is 0 and that J is regular follow e.g. from [BC3] (section 4.3). By

a standard implicit function theorem it follows that the same holds for t 's close enough to 1. Note that also here, if one would replace Δ_k by a line going through the 2'nd and 4'th quadrants, e.g. $\{(x, -x) : x \in \mathbb{R}\}$, things would go wrong. The constant map at 0 would still be a solution but its index would be negative and J would not be regular with respect to it.

It remains to discuss the case when X is non-compact but symplectically convex at ∞ . The proof is very similar to the one for the case when X is closed. Recall that although now X is not compact the objects of $\mathcal{Fuk}^*(X)$ (i.e. the Lagrangians in X) are still assumed to be compact.

The results of Seidel (see Chapter 16e of [Sei3] and [Sei2]) can be used to produce a fibration \mathcal{E} of generic fibre X , in the sense of the definitions in §2.1, in particular this fibration satisfies assumption T_∞ . As in the compact fibre case, we then use the Proposition 2.3.1 to transform the fibration into a tame one that continues to satisfy T_∞ . The proof then pursues just as in the compact case. Indeed, notice that Assumption T_∞ implies that the monodromy is well defined over any path in $\mathbb{C} \setminus \text{Critv}(\pi)$ (and in fact over any path in \mathbb{C} if we restrict the monodromy to “infinity in the fibers”). Similarly, the procedure from page 74 that ensures that the negative gradient flow of $\text{Re}(\pi)$ is defined for all times continues to work in the present setting. Indeed, the fact that the fibers of \mathcal{E} are not compact does not pose any problems because (in the notation of Assumption T_∞) on $\mathcal{E}^\infty \approx \mathcal{E}_{w_0}^\infty \times \mathbb{C}$ this flow is just a translation in the \mathbb{C} -direction. Finally, in what concerns the Floer and perturbation data we use as in §3.3.5 almost complex structures that are split at ∞ as $i \oplus J_0$ with J_0 compatible with the symplectic convexity of (the end) of X . \square

4.5.7. Proof of Proposition 4.5.3. We now explain how to modify the proof of Proposition 4.5.1 under the assumptions of Proposition 4.5.3, namely that X is itself the total space of a tame Lefschetz fibration $\pi_X : X \rightarrow \mathbb{C}$ as described in §4.5.2. Denote by (N, ω) the generic fibre of π_X which is compact or symplectically convex at infinity.

As in the the proof of Proposition 4.5.1, we again construct a Lefschetz fibration $\pi_{\mathcal{E}} : \mathcal{E} \rightarrow \mathbb{C}$ with fiber over w_0 being X . As before, the fibration \mathcal{E} can be assumed to satisfy Assumption T_∞ as well as the other assumptions in §2.1. By applying to this fibration the same procedure as in the proof of Proposition 2.3.1 we may further assume that this fibration is also tame.

In what concerns the Fukaya category $\mathcal{Fuk}^*(\mathcal{E})$ of \mathcal{E} , by inspecting the proof of Proposition 4.5.1, we see that we can actually use here only a smaller category whose objects are cylindrical cobordisms $V \subset \mathcal{E}$ (not necessarily negatively ended) obtained by taking the trail of a given cobordism $Q \subset \mathcal{E}_{w_0} = X$ along a curve $\gamma \subset \mathbb{C} \setminus \text{Critv}(\pi_{\mathcal{E}})$. To avoid confusion denote the Fukaya category involved here by $\mathcal{Fuk}_r^*(\mathcal{E})$ (where r indicates that our objects are restricted as above). Notice that later in the proof we apply certain isotopies (e.g. the negative gradient flow

of $\text{Re}(\pi_{\mathcal{E}})$ to these cobordisms that might not keep them everywhere cylindrical. However, as we shall see below, this is not a problem since for that stage of the proof we do not need the entire Fukaya category anymore but only Floer homology calculations.

Using the notation from Assumption T_{∞} , put $X^{\infty} = \mathcal{E}_{w_0}^{\infty}$ and fix a symplectic identification

$$(56) \quad \mathcal{E}^{\infty} \approx \mathbb{C}_{\mathcal{E}} \times X^{\infty}.$$

Here $\mathbb{C}_{\mathcal{E}}$ stands for the base of the fibration \mathcal{E} , which is just a copy of \mathbb{C} . The subscript \mathcal{E} is there only in order to emphasize the relation to \mathcal{E} . Denote by $\hat{\pi}_X : \mathcal{E}^{\infty} \rightarrow \mathbb{C}_X$ the projection (on the other copy of \mathbb{C}) induced via (56) by $\pi_X : X^{\infty} \rightarrow \mathbb{C}_X$.

Notice that due to the T_{∞} assumption a cobordism $V \in \mathcal{Ob}(\mathcal{Fuk}_r^*(\mathcal{E}))$ has the property that $V \cap \mathcal{E}^{\infty}$ is a union of finitely many components of the form $\gamma \times l_i \times L_i$ where $\gamma \in \mathbb{C}_{\mathcal{E}}$ is the projection of V onto $\mathbb{C}_{\mathcal{E}}$, l_i is a negative ray in \mathbb{C}_X (of imaginary coordinate i in) and $L_i \subset N$ is a Lagrangian in N . To fix ideas we will call these Lagrangians L_i , *the ends of V in the direction of \mathbb{C}_X* . The important fact to keep in mind is that these ends remain constant along γ . Obviously, there are also the “usual” ends of V that are of the form $l_i \times C_i$ where l_i is a ray (negative or positive) in $\mathbb{C}_{\mathcal{E}}$ and $C_i \subset X$ is a negative-ended cobordism in X . We will refer to these cobordisms C_i as the *ends of V in the direction of $\mathbb{C}_{\mathcal{E}}$* . For each V there are obviously at most two such ends. Notice also that the ends of C_i , itself viewed as cobordism, are Lagrangians in N that coincide with the ends of V in the direction of \mathbb{C}_X .

We now pass to explaining the choices of Floer and perturbation data required to define the category $\mathcal{Fuk}_r^*(\mathcal{E})$. We first pick a profile function $h_X : \mathbb{C}_X \rightarrow \mathbb{R}$ such as in §3.3.2 but with the property that the bottlenecks are inside $\pi_X(X^{\infty})$.

Consider $V_1, \dots, V_{k+1} \in \mathcal{Ob}(\mathcal{Fuk}_r^*(\mathcal{E}))$. Let $C^1, \dots, C^s \in \mathcal{Ob}(\mathcal{Fuk}^*(X))$ be the collection of all the ends in the direction of \mathbb{C}_X of the objects V_1, \dots, V_{k+1} . We use the function h_X and the method in §3.3.3 to construct the Floer and perturbation data, associated to C^1, \dots, C^s as objects of the category $\mathcal{Fuk}^*(X)$ associated to the tame Lefschetz fibration $\pi_X : X \rightarrow \mathbb{C}$. We denote all this data by $\mathcal{D}_{V_1, \dots, V_{k+1}}^X$. As described in §3.3, this data consists of particular choices of Hamiltonians on X , that are grouped here in $\mathcal{H}_{V_1, \dots, V_{k+1}}^X$, and almost complex structures on X , grouped in $\mathcal{J}_{V_1, \dots, V_{k+1}}^X$ so that $\mathcal{D}_{V_1, \dots, V_{k+1}}^X = (\mathcal{H}_{V_1, \dots, V_{k+1}}^X, \mathcal{J}_{V_1, \dots, V_{k+1}}^X)$.

Pick a profile function $h_{\mathcal{E}} : \mathbb{C}_{\mathcal{E}} \rightarrow \mathbb{R}$ again as described in §3.3.2. Let γ_i be the projection of V_i onto $\mathbb{C}_{\mathcal{E}}$. Now modify $h_{\mathcal{E}}$, away from the region of the bottlenecks, in such a way that the new function $h_{V_1, \dots, V_{k+1}}$ conserves the same bottlenecks as $h_{\mathcal{E}}$ and, additionally, $(\phi_1^{h_{V_1, \dots, V_{k+1}}})^{-1}(\gamma_i)$ is transverse to γ_j for all i, j . Now define a new set of Hamiltonians, this time defined on $\mathbb{C}_{\mathcal{E}} \times X$ as follows: $\mathcal{H}'_{V_1, \dots, V_{k+1}} = \{h_{V_1, \dots, V_{k+1}} + H : H \in \mathcal{H}_{V_1, \dots, V_{k+1}}^X\}$.

With these choices, we can describe the constraints on the class of Hamiltonians $\mathcal{H}_{V_1, \dots, V_{k+1}}^{\mathcal{E}}$ defined on \mathcal{E} that are part of the perturbation data $\mathcal{D}_{V_1, \dots, V_{k+1}}^{\mathcal{E}} = (\mathcal{H}_{V_1, \dots, V_{k+1}}^{\mathcal{E}}, \mathcal{J}_{V_1, \dots, V_{k+1}}^{\mathcal{E}})$ that we associate to the family V_1, \dots, V_{k+1} , as required to define $\mathcal{Fuk}_r^*(\mathcal{E})$. There is a compact

set $K_{V_1, \dots, V_{k+1}} \subset \mathbb{C}_{\mathcal{E}}$ away from the bottlenecks of $h_{\mathcal{E}}$ and a compact set $K'_{V_1, \dots, V_{k+1}} \subset \mathcal{E}^{\infty}$ away from the bottlenecks of h_X so that the hamiltonians in $\mathcal{H}_{V_1, \dots, V_{k+1}}^{\mathcal{E}}$ coincide with corresponding Hamiltonians in $\mathcal{H}'_{V_1, \dots, V_{k+1}}$ on the set

$$\mathcal{S}_{V_1, \dots, V_{k+1}} = (\mathcal{E}^{\infty} \setminus K'_{V_1, \dots, V_{k+1}}) \cup \pi_{\mathcal{E}}^{-1}(\mathbb{C}_{\mathcal{E}} \setminus K_{V_1, \dots, V_{k+1}}).$$

It is useful to notice at this point that, because the ends of V_i in the direction of \mathbb{C}_X do not change along γ_i this choice of Hamiltonian perturbations ensures the required transversality at ∞ both in the $\mathbb{C}_{\mathcal{E}}$ direction as well as in the \mathbb{C}_X direction. As the Hamiltonians in $\mathcal{H}_{V_1, \dots, V_{k+1}}^{\mathcal{E}}$ are basically arbitrary perturbations of the Hamiltonians in $\mathcal{H}'_{V_1, \dots, V_{k+1}}$ outside of $\mathcal{S}_{V_1, \dots, V_{k+1}}$ this (together with the choice of almost complex structures as detailed below) is also sufficient to achieve the regularity of the relevant moduli spaces.

The family of almost complex structures $\mathcal{J}_{V_1, \dots, V_{k+1}}^{\mathcal{E}}$ associated to V_1, \dots, V_{k+1} satisfies similar constraints. Namely, over $\mathcal{S}_{V_1, \dots, V_{k+1}}$ they are of the form $i_{\mathcal{E}} \oplus J$ with $J \in \mathcal{J}_{V_1, \dots, V_{k+1}}^X$ but can be perturbed freely, so as to insure regularity, outside of $\mathcal{S}_{V_1, \dots, V_{k+1}}$.

With these choices the compactness results required to define the category $\mathcal{Fuk}_r^*(\mathcal{E})$ are valid. More specifically, all solutions u of the relevant perturbed Cauchy-Riemann equation lie in a prescribed compact subset. The argument is very similar to the one in [BC3]. We consider a hamiltonian $\bar{h} : \mathcal{E} \rightarrow \mathbb{R}$ so that away from $\mathcal{S}_{V_1, \dots, V_{k+1}}$, \bar{h} coincides with $h_{\mathcal{E}} \oplus h_X$. We then use the naturality transformation involving \bar{h} , as summarized in §3.3.4, to turn the solutions u into curves v that are (non-perturbed) J -holomorphic away from $\mathcal{S}_{V_1, \dots, V_{k+1}}$. We then apply the open mapping theorem to the projections $\hat{\pi}_X \circ v$ and $\pi_{\mathcal{E}} \circ v$. To summarize, the arguments for both regularity and compactness of the relevant moduli spaces follow closely the corresponding arguments in [BC3] that are used to set up the Fukaya category of cobordisms in $\mathbb{C} \times M$.

Beyond the definition of $\mathcal{Fuk}_r^*(\mathcal{E})$ an additional remark is in order. A key part of the proof in §4.5.4 uses the Floer homology for the pairs (W, V) , (W, T_{Δ}) and (T_{Δ}, V) . In the course of the proof we apply to W and V the negative and positive gradient flows of $\text{Re}(\pi_{\mathcal{E}})$. While V and W are cylindrical, these flows do not preserve cylindricity. Nevertheless, cylindricity is preserved at infinity in the fiber-direction due to Assumption T_{∞} on \mathcal{E} . Therefore the Floer data can easily be adjusted in this case too by using possibly another compactly supported perturbation to ensure transversality.

With this remark taken into account and with the definition of $\mathcal{Fuk}_r^*(E)$ as above the remainder of the proof proceeds just as in the proof of Proposition 4.5.1.

4.6. The decomposition in Theorem A. To construct this decomposition we start with the proof of Theorem 4.2.1.

4.6.1. *Proof of Theorem 4.2.1.* We assume for the moment that we are in the setting of §4.2. In particular, $\pi : E \rightarrow \mathbb{C}$ is a tame Lefschetz fibration with the properties listed there.

Let $V : \emptyset \rightsquigarrow (L_1, \dots, L_s)$ and consider the Lefschetz fibration $\hat{\pi} : \hat{E} \rightarrow \mathbb{C}$ obtained from E by adding singularities as described in §4.4.2. By Proposition 4.4.5 \hat{E} is strongly monotone. The cobordism V continues to be monotone in \hat{E} and the matching spheres \hat{S}_j are monotone too. Moreover, all these Lagrangians are of monotonicity class $*$. Recall also that by assumption $\dim_{\mathbb{R}} E \geq 4$. Consider now the cobordism

$$V' = \tau_{\hat{S}_m} \circ \tau_{\hat{S}_{m-1}} \circ \dots \circ \tau_{\hat{S}_1}(V) \subset \hat{E}.$$

Given $W \in \mathcal{L}^*(E)$ we rewrite the exact sequence in Proposition 4.5.3 as

$$W = (S \otimes HF(S, W) \rightarrow \tau_S W)$$

and deduce that in $D\mathcal{Fuk}^*(\hat{E})$ we have the following decomposition of V :

$$V \cong (\hat{S}_1 \otimes E_1 \rightarrow \hat{S}_2 \otimes E_2 \rightarrow \dots \rightarrow \hat{S}_m \otimes E_m \rightarrow V'),$$

where

$$(57) \quad E_i = HF(\hat{S}_i, \tau_{\hat{S}_{i-1}} \circ \dots \circ \tau_{\hat{S}_1}(V)) .$$

Notice that in $D\mathcal{Fuk}^*(E)$ we have $T_i \cong (J^{E, \hat{E}})^*(\hat{S}_i)$ where $J^{E, \hat{E}}$ is the inclusion (26) and T_i are the thimbles in the statement of Theorem 4.2.1. Thus, in $D\mathcal{Fuk}^*(E)$ we have the decomposition:

$$(58) \quad V \cong (T_1 \otimes E_1 \rightarrow T_2 \otimes E_2 \rightarrow \dots \rightarrow T_m \otimes E_m \rightarrow V') .$$

By Corollary 4.4.4 we know that inside $D\mathcal{Fuk}^*(E)$ we have:

$$(59) \quad V' \cong (\gamma_s \times L_s \rightarrow \gamma_{s-1} \times L_{s-1} \rightarrow \dots \rightarrow \gamma_2 \times L_2)$$

Splicing together (58) and (59) we obtain:

$$V \cong (T_1 \otimes E_1 \rightarrow \dots \rightarrow T_m \otimes E_m \rightarrow \gamma_s \times L_s \rightarrow \dots \rightarrow \gamma_2 \times L_2)$$

which concludes the proof of Theorem 4.2.1. □

4.6.2. *The decomposition in Theorem A.* We assume the setting from Theorem 4.1.1 (which we recall is just a more precise reformulation of Theorem A) and recall a bit of the necessary background. The fibration $\pi : E \rightarrow \mathbb{C}$ is no longer assumed to be tame. All the singularities of π are included in $\pi^{-1}(S_{x,y})$, $x < 0 < y$ and there is a tame fibration $\pi : E_\tau \rightarrow \mathbb{C}$ that coincides with E over $[x - \frac{7}{2}, y + \frac{7}{2}] \times [-\frac{1}{2}, \infty)$ and is tame outside of a set U that contains $(x - 4, y + 4) \times (-1, \infty)$. Recall also the category $\mathcal{Fuk}^*(E_\tau)$ whose objects are cobordisms (with only negative ends) as in Definition 2.2.3. In particular, these cobordisms have ends that project to the axes $(-\infty, -a_U] \times \{i\} \subset \mathbb{C}$. The constant a_U verifies $-a_U < x - 4$. Recall from §3.4 that the objects of the category $\mathcal{Fuk}^*(E; \tau)$ are uniformly monotone cobordisms $V \subset E$ that are cylindrical outside $S_{x-3, y-3}$ and the operations μ_k of $\mathcal{Fuk}^*(E; \tau)$ are defined by means of the corresponding operations in the category $\mathcal{Fuk}^*(E_\tau)$ associated to the tame fibration E_τ .

The decomposition in Theorem 4.1.1 (and thus that in Theorem A) follows rapidly from that in Theorem 4.2.1. Indeed, recall from §3.4 that we have an inclusion:

$$(60) \quad \mathcal{Fuk}^*(E; \tau) \rightarrow \mathcal{Fuk}^*(E_\tau)$$

that is a quasi-equivalence and which, on objects, is defined by $V \rightarrow \bar{V}$ where \bar{V} is obtained by cutting off the the ends of V along the line $\{x - \frac{7}{2}\} \times \mathbb{R}$ and extending them horizontally by parallel transport in the fibration E_τ . As E_τ is a tame fibration, Theorem 4.2.1 can be applied to it. We deduce decompositions involving two types of curves in the plane, the t_k 's and γ_i 's as in Figure 10. The curves γ_i appearing here are included in the negative quadrant $Q_U^- = (-\infty, -a_U] \times [0, \infty)$ and they are away from U . For reasons that will become clear in a moment, it is convenient to refine the notation for these curves such as to explicitly indicate their dependence on U . Thus we will further denote them by γ_i^U .

The decomposition result that we want to show here - for the statement of Theorem 4.1.1 - applies to $\mathcal{Fuk}^*(E; \tau)$. It again involves the same thimbles T_k associated to the curves t_k as before as well certain “trails” denoted in Theorem 4.1.1 by $\gamma_i L_i$. It is important to notice at this point that the curves γ_i appearing in the statement of Theorem 4.1.1 do not coincide with the γ_i^U 's above - see also Figure 32. Indeed, following the definition in §4.1.1, these curves have image inside $(-\infty, x) \times [\frac{1}{2}, \infty)$ and they “bend” inside $[x - 2, x - 1] \times [1, \infty]$, while γ_i^U is away from U and thus away from $(x - 4, y + 4) \times \mathbb{R}$.

Nonetheless, for $L \in \mathcal{L}^*(M)$ and any curve γ_i consider the cobordism $\overline{\gamma_i L}$ as an object of $\mathcal{Fuk}^*(E_\tau)$. This object is quasi-isomorphic to $\gamma_i^U \times L$ (this can be proved directly, but it also follows immediately from Theorem 4.2.1 itself). As a consequence, we may replace in the decomposition given by Theorem 4.2.1 the objects $\gamma_i^U \times L_i$ by the objects $\overline{\gamma_i L_i}$ and by pulling back the resulting decomposition from $\mathcal{Fuk}^*(E_\tau)$ to $\mathcal{Fuk}^*(E; \tau)$ via the inclusion (60) we obtain the decomposition claimed in Theorem 4.1.1. \square

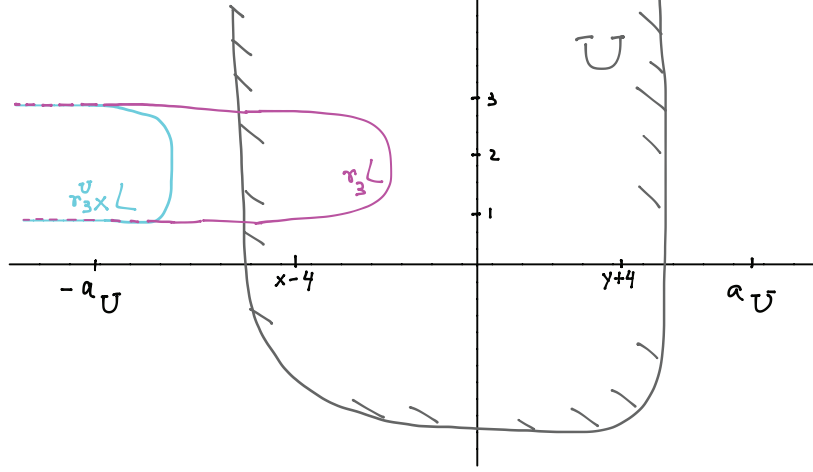


FIGURE 32. The Lagrangian $\gamma_3^U \times L$ is an object in $\mathcal{Fuk}^*(E_\tau)$ but is not cylindrical outside of $[x - 3, y + 3] \times \mathbb{R}$ and thus it not an object in $\mathcal{Fuk}^*(E, \tau)$.

5. MAIN CONSEQUENCES

5.1. From the total space to the fiber and back. We will work in this subsection only with tame Lefschetz fibrations - see Definition 2.2.2. In view of §2.3 this is not restrictive. Thus we assume that $\pi : E \rightarrow \mathbb{C}$ is a Lefschetz fibration which is tame outside of $U \subset \mathbb{C}$ and (M, ω) is the generic fibre. The fibration E has singularities x_1, \dots, x_m of respective critical values v_1, \dots, v_m (assumed to be, for simplicity, $v_k = (k, \frac{3}{2})$). Denote by $O \in \mathbb{C}$ the origin and recall that the fibration E is assumed to be tame over a region that contains O . Connect each critical value v_k to O by a straight segment, and denote by $S_k \in \pi^{-1}(O) = M$ the vanishing cycle associated to that path.

We use the rest of the set-up and notation from §4.2. The results described below are all consequences of Theorem 4.2.1.

5.1.1. *Descent: from decompositions in $D\mathcal{Fuk}^*(E)$ to decompositions in $D\mathcal{Fuk}^*(M)$.*

Corollary 5.1.1. *As in Theorem 4.2.1, let $V \in \mathcal{L}^*(E)$, $V : \emptyset \rightarrow (L_1, \dots, L_s)$. Then there exists an iterated cone decomposition that depends on V and takes place in $D\mathcal{Fuk}^*(M)$:*

$$(61) \quad \begin{aligned} L_1 \cong & (\tilde{\tau}_{2,\dots,m}^{-1} S_1 \otimes E_1 \rightarrow \tilde{\tau}_{3,\dots,m}^{-1} S_2 \otimes E_2 \rightarrow \dots \\ & \rightarrow \tilde{\tau}_{i+1,\dots,m}^{-1} S_i \otimes E_i \rightarrow \dots \rightarrow S_m \otimes E_m \rightarrow L_s \rightarrow L_{s-1} \rightarrow \dots \rightarrow L_2), \end{aligned}$$

where $\tilde{\tau}_{i,\dots,m}$ stands for the composition:

$$\tilde{\tau}_{i,\dots,m} = \tau_{S_i} \circ \tau_{S_{i+1}} \circ \dots \circ \tau_{S_m} .$$

Proof. In this proof it is convenient to consider again the category $D\mathcal{Fuk}_{\frac{1}{2}}^*(E)$ from §4.3. Recall that the difference between this category and $D\mathcal{Fuk}^*(E)$ is that the objects V of the underlying category $\mathcal{Fuk}_{\frac{1}{2}}^*(E)$ are more general cobordisms than those given in Definition 2.2.3 in that the imaginary coordinates of the ends of V are allowed to also be positive half-integers. In other words, V has only negative ends and

$$V \cap \pi^{-1}(Q_U^-) = \coprod_i ((-\infty, -a_U] \times \frac{i}{2}) \times L_i .$$

We now consider curves η_i as in Figure 33.

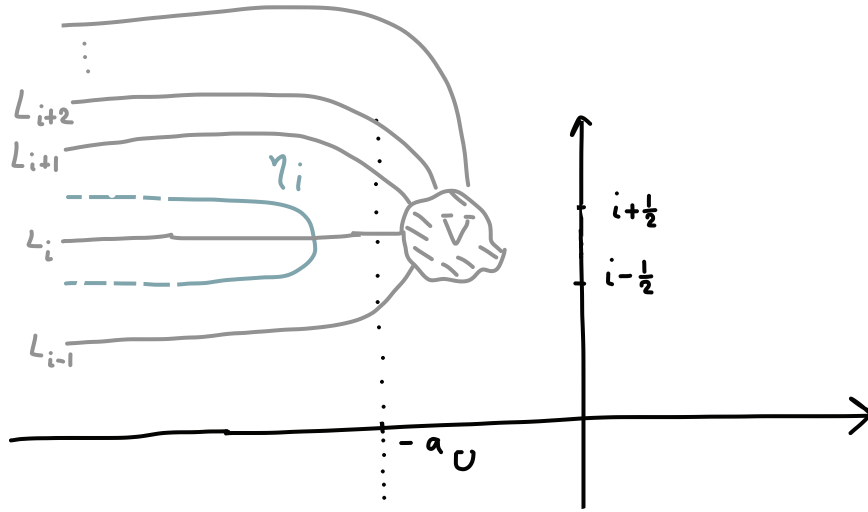


FIGURE 33. The auxiliary curves η_i together with the cobordism $V \in \mathcal{L}^*(E)$.

These curves satisfy

$$\eta_i((-\infty, -1]) = (-\infty, -a_U - 2] \times \frac{2i-1}{2} , \quad \eta_i([1, +\infty)) = (-\infty, -a_U - 2] \times \frac{2i+1}{2}$$

and $\eta_i(\mathbb{R}) \subset Q_U^-$.

As shown in [BC3] §4 there exists an A_∞ -functor:

$$i^{\eta_j} : \mathcal{Fuk}^*(M) \rightarrow \mathcal{Fuk}_{\frac{1}{2}}^*(E)$$

which acts on objects by $L \mapsto \eta_j \times L$. Consider now the pull-back functor:

$$(i^{\eta_j})^* : \text{mod}(\mathcal{Fuk}_{\frac{1}{2}}^*(E)) \rightarrow \text{mod}(\mathcal{Fuk}^*(M)) .$$

Notice that there is a full and faithful embedding $e : \mathcal{Fuk}^*(E) \rightarrow \mathcal{Fuk}_{\frac{1}{2}}^*(E)$. Consider the Yoneda embeddings $\mathcal{Y} : \mathcal{Fuk}^*(E) \rightarrow \text{mod}(\mathcal{Fuk}^*(E))$ and $\mathcal{Y}_{\frac{1}{2}} : \mathcal{Fuk}_{\frac{1}{2}}^*(E) \rightarrow \text{mod}(\mathcal{Fuk}_{\frac{1}{2}}^*(E))$. Let $\mathcal{Y}' : \mathcal{Fuk}^*(E) \rightarrow \text{mod}(\mathcal{Fuk}_{\frac{1}{2}}^*(E))$ be $\mathcal{Y}' = \mathcal{Y}_{\frac{1}{2}} \circ e$. The homology category associated to the

triangular completion $(\text{Image}(\mathcal{Y}'))^\wedge$ of the image of \mathcal{Y}' inside $\text{mod}(\mathcal{Fuk}_{\frac{1}{2}}^*(E))$ is easily seen to be quasi-equivalent to $D\mathcal{Fuk}^*(E)$ (see also §3.1).

For an object $V \in \mathcal{Fuk}^*(E)$ let $\mathcal{M}'_V = \mathcal{Y}'(V)$. Notice that $(i^{\eta_j})^*(\mathcal{M}'_V)$ is precisely the Yoneda module associated to the j -end of V . Thus i^{η_j} takes Yoneda modules to Yoneda modules and given that $H(\text{Image}(\mathcal{Y}'))^\wedge = D\mathcal{Fuk}^*(E)$ we deduce that the functor $(i^{\eta_j})^*$ induces a functor of triangulated categories

$$(62) \quad \mathcal{R}_j : D\mathcal{Fuk}^*(E) \rightarrow D\mathcal{Fuk}^*(M)$$

that we will refer to as the restriction to the j -th end.

The decomposition in the statement is obtained by applying \mathcal{R}_1 to the decomposition in Theorem 4.2.1. Symplectic Picard-Lefschetz theory shows that the end of the thimble T_k is Hamiltonian isotopic to $(\tau_{S_m}^{-1} \circ \tau_{S_{m-1}}^{-1} \circ \tau_{S_{k+1}}^{-1})(S_k) = \tilde{\tau}_{k+1, \dots, m}^{-1} S_k$ and its projection to \mathbb{C} has y -coordinate 1. Clearly, the end of $\gamma_k \times L_k$ over $y = 1$ is L_k for $k \geq 2$ and, similarly, the end of V over $y = 1$ is L_1 . \square

Remark 5.1.2. The functor \mathcal{R}_j from (62) can also be interpreted in a different fashion. We can view it as the triangulated functor induced by an A_∞ -functor $\tilde{\mathcal{R}}_j : \mathcal{Fuk}^*(E) \rightarrow \mathcal{Fuk}^*(M)$ that, on objects, associates to each cobordism $V : \emptyset \rightsquigarrow (L_1, \dots, L_s)$ its j -th end, L_j . It is not difficult to see that, with appropriate choices of auxiliary structures, such a functor is indeed defined and that it induces at the derived level precisely \mathcal{R}_j . At the derived level we also have $\mathcal{R}_j \circ i^{\eta_j} = \text{id}$. Notice also that the pull-back functor

$$\tilde{\mathcal{R}}_j^* : \text{mod}(\mathcal{Fuk}^*(M)) \rightarrow \text{mod}(\mathcal{Fuk}^*(E))$$

takes the Yoneda module $\mathcal{Y}(L)$ to the Yoneda module $\mathcal{Y}(\eta_j \times L) = i^{\eta_j}(L)$.

5.1.2. *Ascent: from $D\mathcal{Fuk}^*(M)$ to the category $D\mathcal{Fuk}^*(E)$.* We assume the same setting as fixed at the beginning of §5.1 and start with some algebraic notation. Let \mathcal{B} be an A_∞ -category (over a given ring \mathcal{A} , e.g. the Novikov ring) and R_1, \dots, R_m a collection of m objects of \mathcal{B} . The following construction is a straightforward extension of the notion of directed A_∞ -category as it appears in [Sei3] (see, in particular, (5m) there).

Consider the ordered set $I_m = \{1, \dots, m\}$ and let \mathbb{N}_{+m} be the disjoint union $\mathbb{N} \cup I_m$ ordered strictly in a way that respects the order of \mathbb{N} and I_m and so that each element in I_m is strictly bigger than any element of \mathbb{N} . We still denote the resulting order relation by \geq . For any two $i, j \in \mathbb{N}_{+m}$ we put $\xi^{i,j} = 1$ if $i \geq j$ and $\xi^{i,j} = 0$ if $i < j$ and we let $\xi^{i_1, i_2, \dots, i_{k+1}} = \xi^{i_1, i_2} \xi^{i_2, i_3} \dots \xi^{i_k, i_{k+1}}$.

We denote by $\mathbb{N}_{+m} \otimes \mathcal{B}$ the unique A_∞ -category with the properties:

- i. The objects of $\mathbb{N}_{+m} \otimes \mathcal{B}$ are couples (i, L) with $i \in \mathbb{N}_{+m}$ and L an object of \mathcal{B} with the constraint that if $i \in I_m$, then $L = R_i$. We will write the couples (i, L) as $i \times L$.

ii. The morphisms of $\mathbb{N}_{+m} \otimes \mathcal{B}$ are defined by:

$$\text{Mor}(i \times L, j \times L') = \xi^{i,j} \text{Mor}_{\mathcal{B}}(L, L')$$

except if $i = j \in I_m$. In this case $\text{Mor}(i \times R_i, i \times R_i) = \mathcal{A}e_{R_i}$. Here e_{R_i} is, by definition, a strict unit in the category $\mathbb{N}_{+m} \otimes \mathcal{B}$.

iii. We denote by

$$\mu_k : \text{Mor}(L_1, L_2) \otimes \text{Mor}(L_2, L_3) \otimes \dots \otimes \text{Mor}(L_k, L_{k+1}) \rightarrow \text{Mor}(L_1, L_{k+1})$$

the multiplications in \mathcal{B} . Consider successive indices $(i_1, i_2, \dots, i_{k+1})$ so that no two successive indexes i_r, i_{r+1} satisfy $i_r = i_{r+1} \in I_m$. Then the multiplications in $\mathbb{N}_{+m} \otimes \mathcal{B}$ are given by:

$$\begin{aligned} \mu'_k : \text{Mor}(i_1 \times L_1, i_2 \times L_2) \otimes \text{Mor}(i_2 \times L_2, i_3 \times L_3) \otimes \dots \otimes \text{Mor}(i_k \times L_k, i_{k+1} \times L_{k+1}) &\rightarrow \\ &\rightarrow \text{Mor}(i_1 \times L_1, i_{k+1} \times L_{k+1}) \\ (63) \quad \mu'_k &= \xi^{i_1, \dots, i_{k+1}} \mu_k \end{aligned}$$

In case for some index r we have $i_r = i_{r+1} \in I_m$, then μ'_k is completely described by the requirement that e_{R_i} be a strict unit: μ'_k vanishes if $k \neq 2$ and $\mu'_2(a, e_{R_i}) = a$, $\mu'_2(e_{R_i}, b) = b$.

The notation $\mathbb{N}_{+m} \otimes \mathcal{B}$ is slightly imprecise as this category actually depends on the choice of objects R_1, \dots, R_m . Moreover, there is obviously an abuse of notation here as $\mathbb{N}_{+m} \otimes \mathcal{B}$ is not a tensor product (there is no addition among the objects etc).

In case the A_∞ -category \mathcal{B} is such that the objects R_i have strict units $e'_{R_i} \in \text{Mor}_{\mathcal{B}}(R_i, R_i)$, then by taking $e_{R_i} = e'_{R_i}$, equation (63) applies without treating separately the case $i_r = i_{r+1} \in I_m$. In general, when the R_i 's do not have strict units, we treat the e_{R_i} 's as formal elements, part of the construction of $\mathbb{N}_{+m} \otimes \mathcal{B}$.

Corollary 5.1.3. *There exists a choice of Lagrangians spheres $R_1, \dots, R_m \in \mathcal{L}^*(M)$ and an equivalence of categories:*

$$\mathcal{I} : D(\mathbb{N}_{+m} \otimes \mathcal{Fuk}^*(M)) \rightarrow D\mathcal{Fuk}^*(E) .$$

Proof. Consider the full and faithful subcategory $\mathcal{F}(E)$ of $\mathcal{Fuk}^*(E)$ whose objects consist of the following two collections:

- i. $\gamma_{i+2} \times L$ with $i \in \mathbb{N}$ and $L \in \mathcal{L}^*(M)$. Here γ_k , $k \geq 2$, are the plane curves defined in §4.1.1 (see also Figure 10).
- ii. the thimbles T_j , $j \in I_m$.

The generation Theorem 4.2.1 combined with the algebraic Lemma 3.34 in [Sei3] implies that there is an equivalence of categories

$$D\mathcal{F}(E) \rightarrow D\mathcal{Fuk}^*(E)$$

induced by the inclusion

$$\mathcal{F}(E) \rightarrow \mathcal{Fuk}^*(E) .$$

We now intend to show the existence of a quasi-equivalence of A_∞ -categories:

$$\Xi : \mathbb{N}_{+m} \otimes \mathcal{Fuk}^*(M) \rightarrow \mathcal{F}(E) .$$

To this end we first pick a specific family of objects R_1, \dots, R_m in $\mathcal{Fuk}^*(M)$. By definition, these objects are the following Lagrangian spheres:

$$R_{m+1-i} := \widetilde{\tau}_{i+1, \dots, m}^{-1}(S_i) , \quad i = 1, \dots, m$$

- see Corollary 5.1.1 for the notation. For $i \in \mathbb{N}$, and $L \in \mathcal{L}^*(M)$, we define $\Xi'(i \times L) = \gamma_{i+2} \times L$. For $i \in I_m$ we define $\Xi'(i \times R_i) = T_{m+1-i}$.

It is not difficult to see - as in the construction of the inclusion functor $\mathcal{I}_{\gamma, h}$ in [BC3], in particular Proposition 4.2.3 there - that by using appropriate choices for the curves γ_i as well as almost complex structures and perturbation data, we can describe the morphisms and higher products in $\mathcal{F}(E)$ by the formulas corresponding to $\mathbb{N}_{+m} \otimes \mathcal{Fuk}^*(M)$. There is however one exception concerning this correspondence and due to it the map Ξ' can not be assumed directly to be a morphism of A_∞ categories: the difficulty comes from the fact that the objects T_j of $\mathcal{F}(E)$ do not, in general, have strict units. However, there is an algebraic argument - Lemma 5.20 in §(5n) in [Sei3] - that applies also to our case with minor modifications and implies that we can replace Ξ' by a true A_∞ functor: $\Xi : \mathbb{N}_{+m} \otimes \mathcal{Fuk}^*(M) \rightarrow \mathcal{F}(E)$ that acts on objects in the same way as Ξ' and so that Ξ is a quasi-equivalence. Clearly, this implies the equivalence of the associated derived categories and the existence of \mathcal{I} . \square

Remark 5.1.4. a. Corollary 5.1.3 extends a result of Seidel in §18 of [Sei3] (see also [Sei4]) which provides a similar description for the subcategory of $D\mathcal{Fuk}^*(E)$ that is generated by the thimbles T_i .

b. It is easy to see by direct calculation that there are inclusions $\mathcal{J}_s : D\mathcal{Fuk}^*(M) \rightarrow D(\mathbb{N}_{+m} \otimes \mathcal{Fuk}^*(M))$ induced by $L \rightarrow (s, L)$ for all $s \in \mathbb{N}$. The compositions $\mathcal{J}'_s = \mathcal{I} \circ \mathcal{J}_s$ have a simple geometric interpretation. Consider the inclusion $i^{\gamma_{s+2}} : \mathcal{Fuk}^*(M) \rightarrow \mathcal{Fuk}^*(E)$ which acts on objects as $L \rightarrow \gamma_{s+2} \times L$. This induces a functor $i^{\gamma_{s+2}} : D\mathcal{Fuk}^*(M) \rightarrow D\mathcal{Fuk}^*(E)$ that coincides with \mathcal{J}'_s .

c. An obvious by-product of this Corollary is that the derived categories $D\mathcal{Fuk}^*(E; \tau)$ from the statement of Theorem 4.1.1 are independent of the choice of tame fibration E_τ up to equivalence. Together with §4.6.2 this concludes the proof of Theorem 4.1.1.

5.2. The Grothendieck group. The purpose of this section is to discuss a variety of consequences of Theorem 4.2.1 in what concerns the morphism Θ from (1) as well as the Grothendieck group itself.

5.2.1. Cobordism groups and the Grothendieck group. We start by defining the appropriate cobordism groups that will be of interest to us here. We will restrict here too the discussion to tame Lefschetz fibrations. Fix such a fibration $\pi : E \rightarrow \mathbb{C}$ that is tame outside $U \subset \mathbb{C}$. Let (M, ω) be the fibre of π at a point $z_0 \in \mathbb{C} \setminus U$. Let $\Omega_{Lag}^*(M; E)$ be the abelian group defined as the quotient of the free abelian group generated by the Lagrangians $L \in \mathcal{L}^*(M)$ -modulo the relations \mathcal{R}_{cob}^E generated by the cobordisms $V : \emptyset \rightsquigarrow (L_1, \dots, L_s)$, $V \in \mathcal{L}^*(E)$ in the sense that to each such V we associate the relation $L_1 + \dots + L_s \in \mathcal{R}_{cob}^E$. Basically, the point of view here is that cobordisms are relators among their ends. As we do not take into account orientations this group is obviously 2-torsion. Notice that all vanishing spheres $S \subset M$ (associated to any path between a critical value of π and z_0) belong to \mathcal{R}_{cob}^E , hence their cobordism class is $0 \in \Omega_{Lag}^*(M; E)$. This follows from the fact that a vanishing sphere is the single end of a cobordism which is a thimble of some path going from one critical value of π to z_0 .

In case $\pi : E \rightarrow \mathbb{C}$ is the trivial fibration (i.e. E splits symplectically as $E = \mathbb{C} \times M$ and $\pi = \text{pr}_{\mathbb{C}}$) we will abbreviate $\Omega_{Lag}^*(M; E)$ by $\Omega_{Lag}^*(M)$.

Remark 5.2.1. a. While we will not explore this issue here, notice that the group $\Omega_{Lag}^*(M; E)$ is the abelianization of a group $\mathcal{G}_{Lag}^*(M; E)$ that is defined as the free *non-abelian* group generated by the $L \in \mathcal{L}^*(M)$ modulo relations $L_1 \cdot L_2 \cdot \dots \cdot L_s$ associated as before to cobordisms $V : \emptyset \rightsquigarrow (L_1, \dots, L_s)$. In other words, in this case we take into account the geometric order of the ends of V .

b. It is easy to adjust the definition of the groups $\Omega_{Lag}^*(-)$ to the case of non-tame fibrations. However, in view of §2.3, all interesting phenomena concerning these cobordism groups are already present in the case of tame fibrations.

Recall the Grothendieck group $K_0(D\mathcal{Fuk}^*(M))$ that is associated to the triangulated category $D\mathcal{Fuk}^*(M)$ as in §3.1. Notice that this group too is 2-torsion because we work in an ungraded setting. We are interested in a quotient of this Grothendieck group that is associated to our tame fibration $\pi : E \rightarrow \mathbb{C}$. To construct it assume x_1, \dots, x_m are the critical points of π and let the corresponding critical values be v_1, \dots, v_m . Then for each i pick a path in \mathbb{C} from v_i to z_0 that does not encounter any other critical value (such as, for instance, the paths t_i in Figure 10). There is an associated thimble to each such path and let Σ_i be the vanishing sphere in $M = \pi^{-1}(z_0)$ that is the end of the thimble from x_i to M . Denote by \mathcal{S}_E the subgroup in $K_0(D\mathcal{Fuk}^*(M))$ that is generated by the spheres Σ_i . Finally, define the quotient:

$$K_0(D\mathcal{Fuk}^*(M); E) = K_0(D\mathcal{Fuk}^*(M)) / \mathcal{S}_E .$$

Corollary 5.2.2. *The group $K_0(D\mathcal{Fuk}^*(M); E)$ does not depend on the choices made in its construction and there exists a morphism of groups:*

$$\Theta^E : \Omega_{Lag}^*(M; E) \rightarrow K_0(D\mathcal{Fuk}^*(M); E)$$

that is induced by $L \rightarrow L$.

This morphism extends the Lagrangian Thom morphism initially constructed in [BC3] and already mentioned at (1)

$$\Theta : \Omega_{Lag}^*(M) \rightarrow K_0(D\mathcal{Fuk}^*(M))$$

Proof. We first discuss the independence of $K_0(D\mathcal{Fuk}^*(M); E)$ of the choices of the vanishing spheres Σ_i . Assume for instance that one of these spheres, say Σ_1 - that is the end of a thimble K_1 that projects to a path k_1 from v_1 to z_0 - is replaced with a sphere Σ'_1 which is the end of a thimble K'_1 , associated to a different path, k'_1 . By the results of Seidel [Sei3], the difference between Σ_1 and Σ'_1 (up to hamiltonian isotopy) can be described as follows: one sphere is obtained from the other by applying a symplectic diffeomorphism ϕ which can be written as word in the elements $\tau_{\Sigma_2}, \dots, \tau_{\Sigma_m}$ (i.e. ϕ is a composition of Dehn twists and their inverses along spheres from the collection $\Sigma_2, \dots, \Sigma_m$). From Seidel's exact triangle as given in Proposition 4.5.1 we see that the subgroups generated, respectively, by $\Sigma_1, \Sigma_2, \dots, \Sigma_m$ and $\Sigma'_1, \Sigma_2, \dots, \Sigma_m$ are the same.

The existence of the morphism Θ^E is now an immediate consequence of the decomposition in Corollary 5.1.1. \square

5.2.2. *The Grothendieck group as an algebraic cobordism group.* We now focus our attention on the category $\mathcal{Fuk}^*(E)$.

For each module $\mathcal{M} \in \mathcal{Ob}(D\mathcal{Fuk}^*(E))$, define $[\mathcal{M}]_j \in \mathcal{Ob}(D\mathcal{Fuk}^*(M))$ by

$$[\mathcal{M}]_j = \mathcal{R}_j(\mathcal{M})$$

where \mathcal{R}_j are the restriction functors defined in the proof of Corollary 5.1.1 (see also Remark 5.1.2). Basically, this extends to all objects in $D\mathcal{Fuk}^*(E)$ the operation that associates to a cobordism V its j -th end. It is easy to see that for all objects \mathcal{M} of $D\mathcal{Fuk}^*(E)$ there are only finitely many non-vanishing $[\mathcal{M}]_j$'s.

We now define another group $\Omega_{Alg}^*(M; E)$, which we call the *algebraic cobordism group*, as the free abelian group generated by all the *isomorphism types* of objects $\in \mathcal{Ob}(D\mathcal{Fuk}^*(M))$ modulo the relations

$$[\mathcal{M}]_1 + [\mathcal{M}]_2 + [\mathcal{M}]_3 + \dots = 0$$

for each $\mathcal{M} \in \mathcal{Ob}(D\mathcal{Fuk}^*(E))$.

The group $\Omega_{Alg}^*(M; E)$ can be viewed as an algebraic cobordism group in the following sense. The generators of this group are the (isomorphism type of) objects of $D\mathcal{Fuk}^*(M)$, thus they

are obtained by completing algebraically the objects of $\mathcal{Fuk}^*(M)$ as in the construction of the derived Fukaya category. Similarly, the relations defining the group are again an algebraic completion - in a similar sense but now involving the categories $\mathcal{Fuk}^*(E)$ and $D\mathcal{Fuk}^*(E)$ - of the relations providing $\Omega_{Lag}^*(M; E)$. By definition, there is an obvious group morphism:

$$q : \Omega_{Lag}^*(M; E) \rightarrow \Omega_{Alg}^*(M; E) .$$

Corollary 5.2.3. *There is a group isomorphism*

$$\Theta_{Alg}^E : \Omega_{Alg}^*(M; E) \rightarrow K_0(D\mathcal{Fuk}^*(M); E)$$

so that $\Theta^E = \Theta_{Alg}^E \circ q$.

Proof. Throughout the proof we abbreviate $K_0 = K_0(D\mathcal{Fuk}^*(M); E)$.

At the level of generators we define Θ_{Alg}^E to be the identity. The surjectivity of Θ_{Alg}^E is clear as well as the relation $\Theta^E = \Theta_{Alg}^E \circ q$. The only two things to check are that this map is well-defined and injective.

To show that Θ_{Alg}^E is well-defined we need to prove that if \mathcal{M} is an object of $D\mathcal{Fuk}^*(E)$, then $\sum_i [\mathcal{M}]_i = 0$ in $K_0(D\mathcal{Fuk}^*(M); E)$. To see this recall that, by the definition of $D\mathcal{Fuk}^*(E)$, there are $V_j \in \mathcal{L}^*(E)$ so that:

$$\mathcal{M} \cong (V_m \rightarrow V_{m-1} \rightarrow \dots \rightarrow V_2 \rightarrow V_1) .$$

By Theorem 4.2.1, in K_0 we have:

$$\sum_i [V_j]_i = 0 , \quad \forall j .$$

Moreover, $\forall i$, we have the following cone decomposition of $[\mathcal{M}]_i$ in $D\mathcal{Fuk}^*(M)$:

$$[\mathcal{M}]_i \cong ([V_m]_i \rightarrow [V_{m-1}]_i \rightarrow \dots \rightarrow [V_2]_i \rightarrow [V_1]_i)$$

because the functor \mathcal{R}_i is triangulated. This means that in K_0 :

$$\sum_i [\mathcal{M}]_i = \sum_{i,j} [V_j]_i = 0 .$$

This concludes the proof of the well-definedness of the map Θ_{Alg}^E .

It remains to show that Θ_{Alg}^E is injective. We start by proving the injectivity in the case when π is trivial and so $E = \mathbb{C} \times M$. We omit E from the notation of Θ_{Alg} in this case and, similarly, we put $\Omega_{Alg}(M) = \Omega_{Alg}(M; \mathbb{C} \times M)$. Assume that

$$\mathcal{M} \rightarrow \mathcal{M}' \rightarrow \mathcal{M}''$$

is an exact triangle of $\mathcal{Fuk}^*(M)$ -modules. The injectivity of Θ_{Alg} follows by constructing for each such triangle an object T in $D\mathcal{Fuk}^*(\mathbb{C} \times M)$ so that $[T]_1 = \mathcal{M}''$, $[T]_2 = \mathcal{M}'$ and

$[T]_3 = \mathcal{M}$. Indeed, this implies that all the relations that are used in the definition of K_0 also appear among the relations that define $\Omega_{Alg}^*(M)$ which means that Θ_{Alg} is invertible.

To construct this object T we proceed as follows. We first recall that, by definition, \mathcal{M}'' is - up to isomorphism - the cone over a module map $f : \mathcal{M} \rightarrow \mathcal{M}'$.

Now recall the A_∞ -category $\mathbb{N} \otimes \mathcal{Fuk}^*(M)$ as in §5.1.2 (notice that now $m = 0$). We first construct an object \tilde{T} of $\mathbb{N} \otimes \mathcal{Fuk}^*(M)$. This consists of two steps. First, for each $\mathcal{Fuk}^*(M)$ -module \mathcal{N} and each curve γ_i we define a $\mathbb{N} \otimes \mathcal{Fuk}^*(M)$ -module denoted by $\gamma_i \times \mathcal{N}$. On objects $\gamma_j \times L$ we put $(\gamma_i \times \mathcal{N})(\gamma_j \times L) = \xi^{j,i} \mathcal{N}(L)$. The A_∞ -module operations are defined by a direct adaptation of the formulas giving the operations in $\mathbb{N} \otimes \mathcal{Fuk}^*(M)$. The second step is to define a morphism

$$\bar{f} : \gamma_3 \times \mathcal{M} \rightarrow \gamma_2 \times \mathcal{M}' .$$

We then define \tilde{T} by $\tilde{T} = \text{cone}(\bar{f})$. The morphism \bar{f} is induced by f and is given by a formula again perfectly similar to the formula of the multiplication in $\mathbb{N} \otimes \mathcal{Fuk}^*(M)$, but using f instead of μ_k and replacing $\text{Mor}(i_k \times L_k, i_{k+1} \times L_{k+1})$ by $(\gamma_3 \times \mathcal{M})(\gamma_{i_k-2} \times L_{k+1})$ and $\text{Mor}(i_1 \times L_1, i_{k+1} \times L_{k+1})$ by $(\gamma_2 \times \mathcal{M}')(\gamma_{i_1-2} \times L_1)$. We now consider the sequence of functors, the first two being equivalences and the last a full and faithful embedding:

$$(64) \quad D(\mathbb{N} \otimes \mathcal{Fuk}^*(M)) \rightarrow D\mathcal{F}(\mathbb{C} \times M) \rightarrow D\mathcal{Fuk}^*(\mathbb{C} \times M) \rightarrow D\mathcal{Fuk}_{\frac{1}{2}}^*(\mathbb{C} \times M).$$

Here, the A_∞ -category $D\mathcal{F}(\mathbb{C} \times M)$ is defined as in the proof of Corollary 5.1.3. We now use the composition of the functors in (64) to define $[\mathcal{H}]_j = (i^{\eta_j})^*(\mathcal{H})$ for each module \mathcal{H} in $D(\mathbb{N} \otimes \mathcal{Fuk}^*(M))$ - see the proof of Corollary 5.1.1 for the definition of i^{η_j} . We take T to be the image of \tilde{T} by the first two equivalences in (64) and we claim that:

- a. for each object \mathcal{N} in $D\mathcal{Fuk}^*(M)$ we have that $[(\gamma_i \times \mathcal{N})]_j \cong \mathcal{N}$ if $i = j$ or $j = 1$ and is 0 otherwise. Moreover, $(i^{\eta_1})^*(\bar{f}) \cong f$.
- b. $[T]_1 = \mathcal{M}''$, $[T]_2 = \mathcal{M}'$, $[T]_3 = \mathcal{M}$ and $[T]_i = 0$ whenever $i \geq 4$.

Notice that point b concludes the proof for $E = \mathbb{C} \times M$. Given that the equivalences in (64) are triangulated, point b follows directly from a. Thus, it remains to check a. For this we notice that pull-back respects triangles and as each object \mathcal{N} is isomorphic to an iterated cone of objects $L \in \mathcal{Fuk}^*(M)$ it is enough to verify the statement for the Yoneda modules $\gamma_i \times L$, $L \in \mathcal{L}^*(M)$. But for these modules the statement is obvious. The statement for \bar{f} follows in a similar fashion.

We are left to show the more general statement for a Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$ that is not trivial. For this we recall that, for each thimble T_i we have $(i^{\eta_1})^*(T_i) = \tilde{\tau}_{i+1, \dots, m}^{-1} S_i$. (The definition of the spheres S_i appears in §5.) Thus, by the definition of the groups involved, we have a quotient map

$$(65) \quad \Omega_{Alg}^*(M)/\mathcal{S}'_E \rightarrow \Omega_{Alg}^*(M; E) \xrightarrow{\Theta_{Alg}^E} K_0(D\mathcal{Fuk}^*(M); E),$$

where \mathcal{S}'_E is the subgroup generated by the vanishing spheres of π . To conclude the proof of the theorem it is enough to show that the composition of maps in (65) is an isomorphism. Recall that

$$K_0(D\mathcal{F}uk^*(M); E) = K_0(D\mathcal{F}uk^*(M))/\mathcal{S}_E$$

and notice that the isomorphism Θ_{Alg} - associated to the trivial fibration $\mathbb{C} \times M$ - has the property that $\Theta_{Alg}(\mathcal{S}'_E) = \mathcal{S}_E$. Therefore the composition of maps in (65) is an isomorphism and this concludes the proof. \square

5.2.3. *Comparison with ambient quantum homology.* There is an obvious morphism:

$$i : \Omega_{Lag}^*(M) \rightarrow QH(M)$$

that associates to each Lagrangian L its homology class $[L] \in H_n(M; \mathbb{Z}_2) \subset QH(M)$. From the point of view of Corollary 5.2.3 it is natural to expect that i factors through a morphism:

$$i' : \Omega_{Alg}^*(M) \rightarrow QH(M) .$$

This is indeed true as we will see below.

Corollary 5.2.4. *Consider a module $\mathcal{M} \in \mathcal{Ob}(D\mathcal{F}uk^*(M))$. Such a module admits a cone-decomposition (up to quasi-isomorphism)*

$$\mathcal{M} \cong (L_s \rightarrow L_{s-1} \rightarrow \dots \rightarrow L_1) .$$

With this notation, the equation

$$(66) \quad i'(\mathcal{M}) = \sum_j [L_j] \in QH(M)$$

provides a well-defined group morphism

$$i' : \Omega_{Alg}^*(M) \rightarrow QH(M)$$

so that $i = i' \circ q$.

Proof. While this definition of i' seems very simple the fact that i' is a well-defined morphism of groups is somewhat surprising. We only know a proof of this fact which follows from the indirect construction that we give below.

We will write i' as a composition of two morphisms $i' = \tilde{i}' \circ \Theta_{Alg}$ where $\Theta_{Alg} : \Omega_{alg}^*(M) \rightarrow K_0(D\mathcal{F}uk^*(M))$ is the isomorphism in Corollary 5.2.3 and

$$\tilde{i}' : K_0(D\mathcal{F}uk^*(M)) \rightarrow QH(M)$$

is a morphism that is known to experts, see for instance § 5 in [Sei5]. The definition of \tilde{i}' is somewhat subtle so we review it here.

The morphism \tilde{i}' is a composition of morphisms:

$$\begin{aligned} K_0(D\mathcal{Fuk}^*(M)) &\xrightarrow{f_1} K_0(\mathcal{Y}(\mathcal{Fuk}^*(M))^\wedge) \xrightarrow{f_2} \\ &\xrightarrow{f_2} HH_*(\mathcal{Y}(\mathcal{Fuk}^*(M))^\wedge) \xrightarrow{f_3} HH_*(\mathcal{Fuk}^*(M)) \xrightarrow{f_4} QH(M) . \end{aligned}$$

Here, the category $\mathcal{Y}(\mathcal{Fuk}^*(M))$ is the Yoneda image of $\mathcal{Fuk}^*(M)$; $(\mathcal{Y}(\mathcal{Fuk}^*(M))^\wedge)$ is its triangular completion (as A_∞ -category); $HH_*(\mathcal{B})$ is the Hochschild homology of the A_∞ -category \mathcal{B} with values in itself (generally denoted by $HH_*(\mathcal{B}, \mathcal{B})$). The morphisms involved are as follows: f_1 is an obvious isomorphism that reflects the definition of the triangular structure of $D\mathcal{Fuk}^*(M)$, the morphism f_2 sends each module in $\mathcal{M} \in \mathcal{Y}(\mathcal{Fuk}^*(M))^\wedge$ to the Hochschild homology class of its unit endomorphism $e_{\mathcal{M}} \in \text{hom}(\mathcal{M}, \mathcal{M})$. The latter descends to K_0 because, as it follows from Proposition 3.8 in [Sei3], if $\mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}''$ is an exact triangle in a triangulated A_∞ -category \mathcal{A} , then $e_{\mathcal{M}} = e_{\mathcal{M}'} + e_{\mathcal{M}''}$ in $HH_*(\mathcal{A})$. The morphism f_3 comes from the fact that the natural inclusion

$$\mathcal{Fuk}^*(M) \rightarrow \mathcal{Y}(\mathcal{Fuk}^*(M))^\wedge$$

induces an isomorphism in Hochschild homology (this is sometimes referred to as a form of Morita invariance. See [Toe] for the analogous though different context of dg-categories); f_3 is the inverse of this isomorphism. Finally, f_4 is the open-closed map (see for instance [Sei5] where it is defined for in the exact case, the adaptation to the monotone setting is immediate). \square

Remark 5.2.5. Assume that \mathcal{M}' is another module in $D\mathcal{Fuk}^*(M)$ as in the statement of the corollary such that $\mathcal{M}' \cong \mathcal{M}$ and

$$\mathcal{M}' = (L'_r \rightarrow L'_{r-1} \rightarrow \dots \rightarrow L'_1) .$$

The existence of i' then implies that $\sum_j [L'_j] = \sum_k [L_k]$. It is interesting to note that the only way we know to show this fact is through the indirect method contained in the proof of the Corollary.

5.2.4. The periodicity isomorphism (2). In view of Corollary 5.1.3 it is natural to expect that $K_0(D\mathcal{Fuk}^*(E))$ can be calculated in terms of $K_0(D\mathcal{Fuk}^*(M))$. We will give here such a calculation but only in the case when E is the trivial fibration $E = \mathbb{C} \times M$. An analogous statement for non-trivial fibrations is expected to also hold, but would require further algebraic elaboration.

Corollary 5.2.6. *There exists a canonical isomorphism*

$$K_0(D\mathcal{Fuk}^*(\mathbb{C} \times M)) \cong \mathbb{Z}_2[t] \otimes K_0(D\mathcal{Fuk}^*(M))$$

induced by the map that sends $\mathcal{M} \in \text{Ob}(D\mathcal{Fuk}^(\mathbb{C} \times M))$ to $\sum_{i \geq 2} t^{i-2} \otimes \mathcal{R}_i(\mathcal{M})$, where \mathcal{R}_i is the restriction functor from (62).*

Proof. From Corollary 5.1.3 it is enough to show that

$$K_0(D(\mathbb{N} \otimes \mathcal{Fuk}^*(M))) \cong \mathbb{Z}_2[t] \otimes K_0(D\mathcal{Fuk}^*(M)) .$$

To simplify notation we denote $G_1 = K_0(D(\mathbb{N} \otimes \mathcal{Fuk}^*(M)))$ and $G_2 = \mathbb{Z}_2[t] \otimes K_0(D\mathcal{Fuk}^*(M))$. Given a module \mathcal{M} which is an object of $D(\mathbb{N} \otimes \mathcal{Fuk}^*(M))$ we use the composition in (64) to define the restriction modules $[\mathcal{M}]_i$ that are objects of $D\mathcal{Fuk}^*(M)$ and define the sum $\phi(\mathcal{M}) = \sum_{i \geq 2} t^{i-2} \otimes [\mathcal{M}]_i \in G_2$. Because the restriction functors \mathcal{R}_j are triangulated it is easy to see that this map descends to a morphism $\phi : G_1 \rightarrow G_2$. The construction of the modules $\gamma_i \times \mathcal{N}$ in the proof of Corollary 5.2.3, in particular point (a) in the course of that proof, shows that ϕ is surjective. To show that ϕ is injective we construct an inverse $\psi : G_2 \rightarrow G_1$. We define $\psi(t^i \otimes \mathcal{N}) = \gamma_{i+2} \times \mathcal{N}$ for each object in $\mathcal{N} \in D\mathcal{Fuk}^*(M)$, where we have used here the notation from the proof of Corollary 5.1.3. Once we show that ψ is well defined (in other words, that it respects the relations giving K_0) it immediately follows that it is an inverse of ϕ by the point (a) in the proof of Corollary 5.2.3. But again as in the proof of Corollary 5.2.3, namely the construction of \tilde{T} , it is easy to see that the map $\mathcal{N} \mapsto \gamma_i \times \mathcal{N}$ respects triangles. As a consequence, ψ is well defined and this concludes the proof. \square

6. EXAMPLES

The purpose of this section is to exemplify various aspects of the machinery in the paper. We start by making more explicit the structure contained in the writing of the cone-decompositions in Theorem A and exemplify this in the simplest possible setting consisting of cobordisms in \mathbb{C} . We then indicate how the cone-decompositions associated to cobordisms in our previous paper [BC3] are a consequence of the results here. We pursue with some cobordism examples in non-trivial Lefschetz fibrations. We first consider a simple horse-shoe like curve in a Lefschetz fibration with just one critical value and make explicit how Seidel's exact sequence follows by applying our machinery to this case. Finally, and this is the novel and longest part of the section, we discuss real Lefschetz fibrations and their relation to Lagrangian cobordism.

6.1. Unwrapping cone-decompositions. The decompositions provided by Theorem A contain more structure than it appears superficially in the writing:

$$V \cong (T_1 \otimes E_1 \rightarrow T_2 \otimes E_2 \rightarrow \dots \rightarrow T_m \otimes E_m \rightarrow \gamma_s L_s \rightarrow \gamma_{s-1} L_{s-1} \rightarrow \dots \rightarrow \gamma_2 L_2) .$$

Namely, see also §3.1.1, writing

$$V \cong (C_3 \rightarrow C_2 \rightarrow C_1)$$

actually means

$$V \cong \text{cone}(C_3 \xrightarrow{f_3} \text{cone}(C_2 \xrightarrow{f_1} C_1))$$

and the attaching maps f_i as well as the intermediate cones are, of course, crucial in determining the result of the iterated cone.

This point is already in evidence in the simplest setting to which can be applied the machinery of the paper: cobordisms in \mathbb{C} without any positive ends (and with the negative ends having integral imaginary coordinates). Obviously, these cobordisms are simply disjoint unions of circles and arcs diffeomorphic to \mathbb{R} with horizontal ends pointing in the negative direction. Notice that due to the uniform monotonicity condition all circles have to enclose the same area. At the same time, circles do not play a significant role here since they have vanishing quantum homology and thus they are not seen by Floer and Fukaya category machinery.

Consider two Lagrangians V and V' as in Figure 34 below.

Namely, V consists of two connected components: V_0 and V_1 with V_0 an arc with ends at height 2 and 6 and V_1 an arc with ends at height 3 and 5; V' has also two components V'_0 an arc with ends at height 2 and 3 and V'_1 again an arc with ends at height 5 and 6. It is easy to see that V and V' are the results of the two types of surgery on the Lagrangians W and W' in the middle part of Figure 34. This means, in particular, as seen in [BC2] that V and

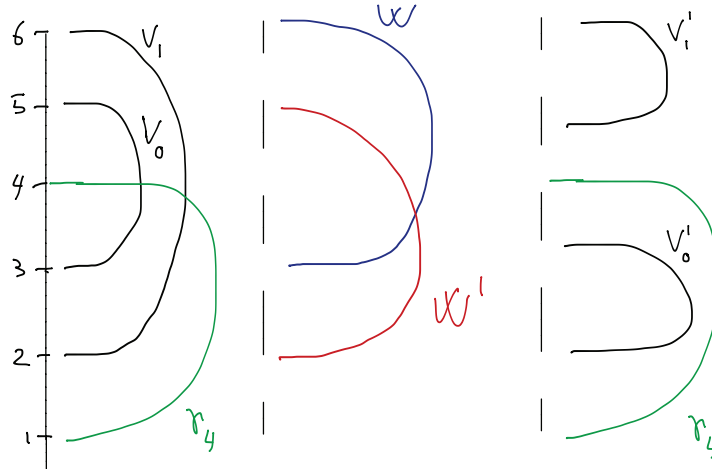


FIGURE 34. The planar cobordisms $V = V_0 \cup V_1$ and $V' = V'_0 \cup V'_1$. They are obtained through the two types of surgery on W and W' . We have $HF(\gamma_4, V) \neq HF(\gamma_4, V')$.

V' are themselves Lagrangian cobordant.

Theorem A applied to V and V' produces decompositions that, formally, in the writing of the statement of that Theorem both look as:

$$(\gamma_6 \rightarrow \gamma_5 \rightarrow \gamma_3 \rightarrow \gamma_2) .$$

However, it is easy to see that V and V' are not isomorphic objects in $D\mathcal{Fuk}^*(\mathbb{C})$. Indeed, $HF(\gamma_4, V) \neq 0$ but $HF(\gamma_4, V') = 0$ and it is an easy exercise to see that the actual two cone decompositions associated to V and V' by Theorem A are different: the intermediate cones and the relevant attaching maps are not the same.

Other examples relevant in this context are associated to elementary Lagrangian cobordisms $W : Q \rightsquigarrow Q$, $W \subset \mathbb{C} \times M$ (here (M, ω) is our fixed symplectic manifold). Examples of such cobordisms are provided by Lagrangian suspension. To such a W we can easily associate a cobordism $V : \emptyset \rightsquigarrow (\emptyset, Q, Q)$. This can be done by first translating W by using $(z, x) \rightarrow (z + i, x)$ and then bending the positive end to the right and extending it to $-\infty$ so that it has height 3. The ends of V have heights 2 and 3 - as in Figure 35. Of course, the simplest

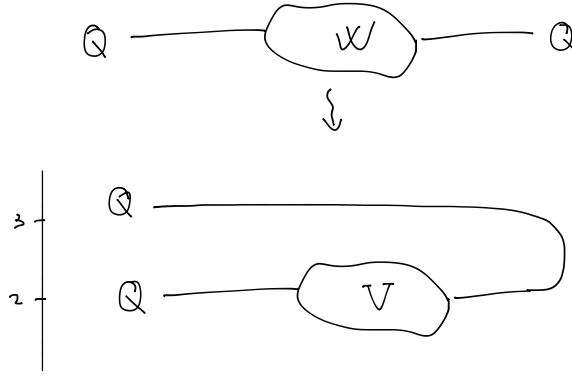


FIGURE 35. The cobordism V is obtained by bending the positive end of the elementary cobordism $W : Q \rightsquigarrow Q$.

such example, V_0 , is associated to the trivial cobordism $W_0 = \mathbb{R} \times \{0\} \times Q$.

The first remark for this class of examples is that all such V 's are isomorphic in $D\mathcal{Fuk}^*(\mathbb{C} \times M)$ to V_0 . The reason is that from Theorem A we have a decomposition:

$$V \cong \text{cone}(\gamma_3 \times Q \xrightarrow{\bar{\varphi}_V} \gamma_2 \times Q) .$$

The morphism $\bar{\varphi}_V$ can be identified with a class $\varphi_V \in HF(Q, Q)$ which is given by the image of the fundamental class $[Q] \in HF(Q, Q)$ under the morphism φ defined as in Equation (40) - see also Figure 24 (of course, in our discussion here the fibration is trivial so that both ends of V in Figure 24 are equal to Q). Moreover, φ_V is an invertible element (see also [BC2]). As a consequence, the cone over $\bar{\varphi}_V$ is easily identified with the cone over $\bar{\varphi}_{V_0}$, where $\varphi_{V_0} = [Q]$.

In short, the two decompositions are isomorphic as in the diagram below

$$(67) \quad \begin{array}{ccccc} \gamma_3 \times Q & \xrightarrow{\bar{\varphi}_V} & \gamma_2 \times Q & \longrightarrow & V \\ id \downarrow & & \downarrow \varphi_V^{-1} & & \downarrow \\ \gamma_3 \times Q & \xrightarrow{\bar{\varphi}_{V_0}} & \gamma_2 \times Q & \longrightarrow & V_0 \end{array}$$

but they are not identical.

6.2. Decompositions in $D\mathcal{Fuk}^*(M)$ induced from cobordisms in $\mathbb{C} \times M$. Let V' be a cobordism $V' : \emptyset \rightsquigarrow (L_1, \dots, L_k)$, $V' \subset (\mathbb{C} \times M, \omega_0 \oplus \omega)$. Theorem A and its Corollary 5.1.1 associate to V' a cone decomposition

$$(68) \quad L_1 \cong (L_k \rightarrow L_{k-1} \rightarrow \dots \rightarrow L_2)$$

At the same time, the machinery in [BC3] applies to cobordisms $V'' : L \rightsquigarrow (L_1, \dots, L_k)$ and associates to such a V'' another cone decomposition:

$$(69) \quad L \cong (L_k \rightarrow L_{k-1} \rightarrow \dots \rightarrow L_1)$$

We want to briefly remark here that the decomposition (69) is a consequence of (68). By elementary manipulations, to see this it is sufficient to consider a cobordism $V : \emptyset \rightsquigarrow (L_2, L_3, \dots, L_k)$ without positive ends and with the first negative end, L_1 , also empty and show that the cone decompositions (69) and (68), both associated to V , coincide.

For this, notice that, by following the proofs of Theorem A and Corollary 5.1.1, the cone decomposition (68) is deduced from the following exact sequences of $\mathcal{Fuk}^*(M)$ modules:

$$(70) \quad W'_{E,i-1}(r \times -) \rightarrow W'_{E,i}(r \times -) \rightarrow \mathcal{Y}(L_i) .$$

Here $W'_{E,i}$ are the $\mathcal{Fuk}^*(\mathbb{C} \times M)$ modules that are introduced at the Step 3 of the proof of Proposition 4.3.1, r is the horizontal line $r = \mathbb{R} \times \{1\}$ and $-$ stands for a variable $Y \in \mathcal{Ob}(\mathcal{Fuk}^*(M))$. The first map in (70) is an inclusion and the second a quotient. There is a slight abuse here as cobordisms of type $r \times Y$ have obviously a positive end by contrast to the objects considered in most of this paper, still the modules $W'_{E,i}(r \times -)$ are well defined. Indeed, as explained at the Step 3 of the proof of Proposition 4.3.1, $W'_{E,i-1}(r \times Y)$ is generated by the intersection points of $r \times Y$ with the first i branches of W' where W' is, in our case, obtained from V by a Hamiltonian isotopy that keeps its ends fixed and moves the non-cylindrical part of V in the lower half-plane - see, for instance, Figure 15. By inspecting [BC3], we see that the cone decomposition (69) follows from exact sequences of $\mathcal{Fuk}^*(M)$ modules:

$$\mathcal{M}_{V,i-1} \rightarrow \mathcal{M}_{V,i} \rightarrow \mathcal{Y}(L_i) .$$

For the description of these modules see Figure 4 and Equation (4) in [BC3]. It immediately follows that $\mathcal{M}_{V,i} = W'_{E,i}(r \times -)$ and thus (68) and (69) are identified.

6.3. A simple cobordism in a Lefschetz fibration with a single critical point. Consider a Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$ of fibre (M, ω) and with a single singularity x_1 of critical value v_1 . We assume that the fibration is tame outside a set $U \subset \mathbb{C}$ as in Figure 36 and we consider a cobordism $V \subset E$ that projects to the curve $\gamma \in \mathbb{C}$. As in the picture this curve turns once around v_1 . Are also pictured there the curves γ_2 and t_1 that appear in the statement of Theorem A as well as the “mirror” singularity x'_1 and the matching sphere \hat{S}_1 that appear in the proof of this theorem (see §4.6).

By the relation between the Dehn twist and the monodromy of Lefschetz fibrations, the ends of V are so that if the first end of V is the Lagrangian $L \subset M$, then the second end is $\tau_S L$ for S an appropriate vanishing sphere associated to x_1 , this can be taken to be the sphere over the end of the curve t_1 .

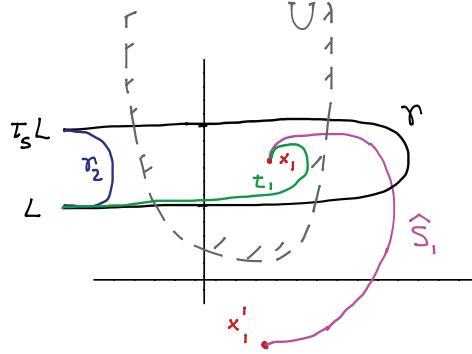


FIGURE 36. The curves γ , γ_2 , t_1 , the region U outside which the fibration $\pi : E \rightarrow \mathbb{C}$ is tame and the matching sphere \hat{S}_1 that is included in the extended fibration $\hat{\pi} : \hat{E} \rightarrow \mathbb{C}$.

Theorem A applied to V shows that:

$$(71) \quad V \cong \text{cone}(T_1 \otimes E_1 \rightarrow \gamma_2 \times \tau_S L)$$

where, as in (57), $E_1 = HF(\hat{S}_1, V)$. In this case we easily see that $HF(\hat{S}_1, V) \cong HF(S, L)$. By applying the restriction functor \mathcal{R}_1 to the equation (71) we obtain

$$L \cong \text{cone}(S \otimes HF(S, L) \rightarrow \tau_S L)$$

which is just another way to express Seidel’s exact triangle from Proposition 4.5.1.

It is instructive to briefly discuss the case when the intersection between S and L is a single point. In this case consider a thimble \hat{T}_1 that is included in the initial fibration $\pi : E \rightarrow \mathbb{C}$ and covers the curve that is given by the projection of \hat{S}_1 in Figure 36 but extended horizontally to $-\infty$. (there is no added singularity x'_1 in this case). This thimble intersects V in a single point and one can surger V and \hat{T}_1 at this point. The resulting manifold $\hat{V} = \hat{T}_1 \# V$ is monotone

and has cylindrical ends S , L and $\tau_S L$. Moreover, by the same arguments as in §4.4, \hat{V} can be Hamiltonian isotoped (with compact support) away from U . That means that \hat{V} can be actually regarded as a cobordism embedded in $\mathbb{C} \times M$ and thus the decomposition result from [BC3] (that applies to cobordisms in $\mathbb{C} \times M$) implies already the existence of the exact triangle $L \cong (S \rightarrow \tau_S L)$. This argument applies as well when the initial cobordism V is more general than the one discussed till now but again under the restriction that \hat{T}_1 intersects V (transversely) in a single point.

Coming back to our V , pictured in Figure 36, there is yet another equivalent approach to produce a cobordism \hat{V} with the properties mentioned above that is possibly even more direct. This is pictured in Figure 37. In this case, we consider a thimble T' that goes horizontally

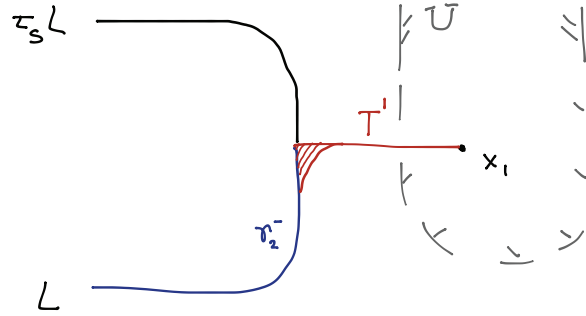


FIGURE 37. Y -surgery between $\gamma_2^- \times L$ and the thimble T' in case L and S have a single intersection point.

towards $-\infty$ starting from x_1 and we do a Y -surgery in a single point between T' and $\gamma_2^- \times L$. Here γ_2^- is the first half of the curve γ_2 and Y -surgery is the construction of the trace of the surgery as Lagrangian cobordism as described in [BC2] §6.1. We can then cut T' outside of U and thus obtain another cobordism which can be regarded as embedded in $\mathbb{C} \times M$. Moreover the latter cobordism will have S , L and $\tau_S L$ as its ends. Finally, it is useful to note that in case the number of intersection points of L and S is at least two, both constructions above fail. In both cases, it is still possible to do an iterated surgery with a number of thimbles equal to the number of intersection points between L and S , basically by the same method as described in §4.4.3. However when using these copies either cylindricity at infinity is lost or the resulting manifold, after surgery, is no longer embedded but only immersed. As an example, if we perform the Y -surgery in the case when there are two intersection points and project the resulting manifold \hat{V} onto \mathbb{C} the image of \hat{V} is as in Figure 38: the thimble T' can be conserved as before - its projection is in red - but the additional copy of it, T'' , will project as the green dotted region there, and it is not clear how to obtain a cobordism (which

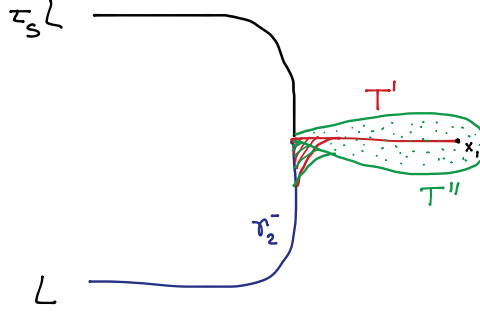


FIGURE 38. Iterated Y -surgery with two thimbles T' and a copy T'' of T' . The projection of T'' fills the green dotted area.

is cylindrical at ∞) when passing to $\mathbb{C} \setminus U$. As a last remark, this \hat{V} , or a small perturbation thereof, can also be viewed as obtained by stretching V in the direction of $-\nabla \text{Re}(\pi)$.

6.4. Changes of generators. The generators appearing in Theorem A, in particular, the T_i 's are not always the most convenient for calculations even if they appear naturally in our proof. It is however easy to change generators in case a different choice is preferable. We exemplify this in the case of one Lefschetz fibration which we assume to fit the setting of Theorem 4.2.1 and with only three critical points, of critical values v_1, v_2, v_3 . In particular, $m = 3$.

We consider two families of thimbles $T_i, T'_i, i = 1, 2, 3$, that are like in the statement of Theorem A and such that the T_i 's cover curves t_i and the T'_i 's cover curves t'_i as in in Figure 39.

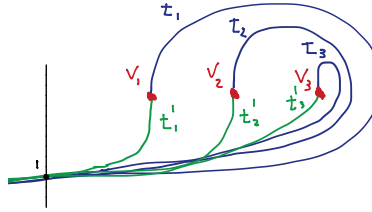


FIGURE 39. The projections t_i and respectively t'_i , of the thimbles T_i , respectively $T'_i, i = 1, 2, 3$ associated to the critical points x_1, x_2, x_3 of critical values v_1, v_2, v_3 .

It is easy to see that by applying Theorem A to the thimbles T'_i we obtain first $T'_3 \cong T_3$. Further, $T'_2 \cong \text{cone}(T_2 \rightarrow T_3 \otimes E_3^2)$, with $E_3^2 = HF(\hat{S}_3, \tau_{\hat{S}_2} \tau_{\hat{S}_1} T'_2)$. Notice also $\tau_{\hat{S}_1} T'_2 = T'_2$ and $\tau_{\hat{S}_2} T'_2$ is just the one point surgery between \hat{S}_2 and T'_2 . It follows $E_3^2 \cong HF(S_3, S_2)$ where S_i are vanishing spheres associated to the singularity x_i (inside a fixed fibre $(M, \omega) = \pi^{-1}(z_0)$). Thus

$$T_2 \cong (T'_3 \otimes HF(S_3, S_2) \rightarrow T'_2) .$$

Similarly, $T'_1 \cong \text{cone}(T_1 \rightarrow T_2 \otimes E_2^1 \rightarrow T_3 \otimes E_3^1)$ and we can again estimate: $E_2^1 = HF(\hat{S}_2, \tau_{\hat{S}_1} T'_1) \cong HF(S_2, S_1)$, $E_3^1 = HF(\hat{S}_3, \tau_{\hat{S}_2} \tau_{\hat{S}_1} T'_1)$. Thus we get:

$$T_1 \cong (T'_3 \otimes HF(S_3, S_2) \otimes HF(S_2, S_1) \rightarrow T'_2 \otimes HF(S_2, S_1) \rightarrow T'_3 \otimes E_3^1 \rightarrow T'_1) .$$

This expression can be further simplified. For instance, the second and third terms can be switched because $\text{hom}(T'_2, T'_3)$ is acyclic (i.e. $HF(T'_2, T'_3) = 0$). In conclusion, we can write

$$T_1 \cong (T'_3 \otimes E'_3 \rightarrow T'_2 \otimes E'_2 \rightarrow T'_1)$$

for appropriate \mathcal{A} -modules E'_3, E'_2 . Using these arguments the decompositions given by Theorem A can be re-written in the generators T'_i : the sequence $(T_1 \otimes E_1 \rightarrow \dots T_3 \otimes E_3)$ inside the cone-decomposition provided by that theorem will be replaced by $(T'_3 \otimes G_3 \rightarrow T'_2 \otimes G_2 \rightarrow T'_1 \otimes G_1)$ for appropriate modules G_i .

The manipulations above can be extended to fibrations with more than three singularities in a straightforward way. The main difficulty in making these changes of generators explicit is in determining the modules G_i . In this respect, it is useful to note that there exists an alternative proof of the decompositions in Theorem A that avoids the geometric disjunction step contained in §4.4 and implements iteratively the stretching argument in §4.5 to the case of more singularities. While this method becomes quite involved for more than a few singularities, it offers sometimes a more direct way to estimate the relevant modules for specific generating families of thimbles.

6.5. Real Lefschetz fibrations. Real Lefschetz fibrations have recently been studied from the topological and real algebraic geometry viewpoints (see e.g. [DS, Sal1, Sal2, Sal3]). Lagrangian cobordism is naturally related to this notion and we describe this relationship in the first subsection below. We then pursue with a construction of such fibrations and, in the last subsection, with a concrete example.

6.5.1. Lagrangian cobordism and real Lefschetz fibrations. Let $\pi : E \rightarrow \mathbb{C}$ be a Lefschetz fibration endowed with a symplectic structure Ω , as in Definition 2.1.1. Denote by (M, ω) the general fiber of (E, Ω) . Let $c_E : E \rightarrow E$ be an anti-symplectic involution, i.e. $c_E^* \Omega = -\Omega$ and $c_E \circ c_E = \text{id}$. Assume further that c_E covers the standard complex conjugation $c_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{C}$, namely $\pi \circ c_E = c_{\mathbb{C}} \circ \pi$. Denote by $V = \text{Fix}(c_E)$ the fixed point locus of c_E . Note that the projection $\pi(V)$ of V to \mathbb{C} is a subset of \mathbb{R} . The following proposition shows that V is a Lagrangian cobordism and also gives a criterion for its monotonicity.

Proposition 6.5.1. *Under the above assumptions V is a Lagrangian cobordism with at most one positive end and at most one negative one (but possibly without any ends at all). Its projection $\pi(V) \subset \mathbb{R}$ is of the form $\cup_{j \in \mathcal{S}} \bar{I}_j$, where \mathcal{S} is a subset of the set of connected*

components of $\mathbb{R} \setminus \text{Critv}(\pi)$, I_j stands for the path connected component corresponding to j and \bar{I}_j is the closure of I_j . Thus $\partial\pi(V)$ is a subset of $\text{Critv}(\pi) \cap \mathbb{R}$.

Moreover, for every $z \in \mathbb{R} \setminus \text{Critv}(\pi)$ the part of V lying over z , $V_z := E_z \cap V$, coincides with the fixed point locus of the anti-symplectic involution $c_E|_{E_z}$ hence is either empty or a smooth Lagrangian submanifold of E_z (possibly disconnected). In particular, the Lagrangians corresponding to the ends of V (if they exist) are real with respect to restriction of c_E to the regular fibers over the real axis at $\pm\infty$.

If (E, Ω) is a monotone symplectic manifold then V is a monotone Lagrangian submanifold of E . Further, denote by $c_1^{\min}(E)$ the minimal Chern number on spherical classes in E and by N_V the minimal Maslov number of V . If $c_1^{\min}(E)$ is odd then $c_1^{\min}(E)|N_V$, and if $c_1^{\min}(E)$ is even then $\frac{1}{2}c_1^{\min}(E)|N_V$.

If $\dim_{\mathbb{C}} M \geq 2$ and (M, ω) is monotone then (E, Ω) is monotone too and $c_1^{\min}(E) = c_1^{\min}(M)$, hence V is a monotone Lagrangian cobordism.

Proof. That V is a (smooth) Lagrangian submanifold follows from it being the fixed point locus of an anti-symplectic involution.

We now show that V is a cobordism and prove the other statements about the projection $\pi(V)$. Since V is Lagrangian, $D\pi_x|_{T_x V} \rightarrow \mathbb{R}$ vanishes iff $x \in \text{Crit}(\pi)$ (see e.g. Chapter 16 of [Sei3]). It follows that $\pi(V) \setminus \text{Critv}(\pi)$ is an open subset of \mathbb{R} and all the points in this subset are regular values of the projection $\pi|_V : V \rightarrow \mathbb{R}$. By construction $V \subset E$ is a closed subset. Therefore if $I \subset \mathbb{R} \setminus \text{Critv}(\pi)$ is a connected component and $\pi(V) \cap I \neq \emptyset$ then $I \subset \pi(V)$. Next, notice that since V is Lagrangian it is invariant with respect to parallel transport along any intervals $I \subset \pi(V) \setminus \text{Critv}(\pi)$.

The statements about $V_z = \text{Fix}(c_E|_{E_z})$ follow directly from the definitions.

We now address the monotonicity of V . This follows from spherical monotonicity of (E, Ω) , by a standard reflection argument based on the existence of the anti-symplectic involution c_E and the fact that $V = \text{Fix}(c_E)$.

Finally, it remains to prove the statement relating the spherical monotonicity of (M, ω) with that of E . Let $E_{z_0} \subset E$ be a smooth fiber endowed with the symplectic structure induced by Ω (so that (M, ω) is symplectomorphic to E_{z_0}). Assume that $\dim_{\mathbb{C}} E_{z_0} \geq 2$ and that E_{z_0} is monotone. It is easy to see that the inclusion, $\pi_2(E_{z_0}) \rightarrow \pi_2(W)$ is surjective and this implies the monotonicity statement. \square

In the next subsection we will show how to construct real Lefschetz fibrations out of Lefschetz pencils arising in real algebraic geometry.

6.5.2. Constructing real Lefschetz fibrations. Let X be a smooth complex projective variety endowed with a real structure, namely an anti-holomorphic involution $c_X : X \rightarrow X$. Let \mathcal{L} be a very ample line bundle on X and assume further that it is endowed with a real structure

compatible with c_X . By this we mean an anti-holomorphic involution $c_{\mathcal{L}} : \mathcal{L} \rightarrow \mathcal{L}$ covering c_X , i.e. $\text{pr} \circ c_{\mathcal{L}} = c_X \circ \text{pr}$, where $\text{pr} : \mathcal{L} \rightarrow X$ is the bundle projection.

Denote by $H^0(\mathcal{L})$ the space of holomorphic sections of \mathcal{L} and by $\mathbf{P} := \mathbb{P}(H^0(\mathcal{L}))^*$ the projectivization of its dual (which can also be thought of as the space of hyperplanes in $H^0(\mathcal{L})$). We denote by $\mathbf{P}^* := \mathbb{P}H^0(\mathcal{L})$ the projectivization of the space of sections itself. Note that \mathbf{P}^* is the dual projective space of \mathbf{P} , hence the notation.

The real structure of \mathcal{L} induces a real structure c_H on $H^0(\mathcal{L})$ defined by $c_H(s) = c_{\mathcal{L}} \circ s \circ c_X$. Denote by $H_{\mathbb{R}}^0(\mathcal{L}) \subset H^0(\mathcal{L})$ the space of real sections of \mathcal{L} (i.e. sections s with $c_H(s) = s$). The real structure c_H descends to real structures on \mathbf{P}^* and \mathbf{P} which, by abuse of notation, we continue to denote both by c_H . The fixed point locus of c_H on \mathbf{P} will be denoted by $\mathbf{P}_{\mathbb{R}}$ and that on \mathbf{P}^* by $\mathbf{P}_{\mathbb{R}}^*$.

Consider now the projective embedding defined using the sections of \mathcal{L} , $X \hookrightarrow \mathbf{P}$. This embedding is real in the sense that it commutes with (c_X, c_H) . Furthermore, there is an isomorphism between \mathbf{P} and $\mathbb{C}P^N$ which sends c_H to the standard real structure $c_{\mathbb{C}P^N}$ of $\mathbb{C}P^N$ (hence $\mathbf{P}_{\mathbb{R}}$ is sent under this isomorphism to $\mathbb{R}P^N$). We fix once and for all such an isomorphism. Denote by $\omega_{\mathbb{C}P^N}$ the standard symplectic structure of $\mathbb{C}P^N$ normalized so that the area of $\mathbb{C}P^1$ is 1. Since $c_{\mathbb{C}P^N}$ is anti-symplectic with respect to $\omega_{\mathbb{C}P^N}$ the previously mentioned isomorphism yields a Kähler form $\omega_{\mathbf{P}}$ on \mathbf{P} and therefore also a Kähler form ω_X on X so that c_X is anti-symplectic with respect to ω_X .

Let $\Delta(\mathcal{L}) \subset \mathbf{P}^*$ be the discriminant locus (a.k.a. the dual variety of X), which by definition is the variety consisting of all section $[s] \in \mathbf{P}^*$ (up to a constant factor) which are somewhere *non-transverse* to the zero-section. Denote by $\Delta_{\mathbb{R}}(\mathcal{L}) = \Delta(\mathcal{L}) \cap \mathbf{P}_{\mathbb{R}}^*$ its real part.

Let $\ell \subset \mathbf{P}^*$ be a line which is invariant under c_H and intersects $\Delta(\mathcal{L})$ only along its smooth strata and transversely. Fix an isomorphism $\ell \approx \mathbb{C}P^1$ and endow ℓ with a standard Kähler structure ω_{ℓ} normalized so that its total area is 1. Consider the symplectic manifold $\ell \times X$ endowed with the symplectic structure $\omega_{\ell} \oplus \omega_X$. For every $\lambda \in \mathbf{P}^*$ denote by $\Sigma^{(\lambda)} = s^{-1}(0) \subset X$ the zero locus corresponding to a section s representing λ . (The varieties $\Sigma^{(\lambda)}$ are sometimes called hyperplane sections since they can also be viewed as the intersection of the image of X in \mathbf{P} with linear hyperplanes.) Note that for all $\lambda \notin \Delta(\mathcal{L})$, the variety $\Sigma^{(\lambda)}$ is smooth. We endow these varieties with the symplectic structure induced from ω_X . The complement of the discriminant, $\mathbf{P}^* \setminus \Delta(\mathcal{L})$, is path connected (since $\Delta(\mathcal{L})$, being a proper complex subvariety of \mathbf{P}^* , has real codimension ≥ 2). Therefore all the symplectic manifolds $\Sigma^{(\lambda)}$, $\lambda \in \mathbf{P}^* \setminus \Delta(\mathcal{L})$, are mutually symplectomorphic.

For every $\lambda \in \mathbf{P}_{\mathbb{R}}^* \setminus \Delta_{\mathbb{R}}(\mathcal{L})$ the manifold $\Sigma^{(\lambda)}$ has a real structure induced by c_X . Denote its real part by $\Sigma_{\mathbb{R}}^{(\lambda)}$. We stress that *in contrast to $\mathbf{P}^* \setminus \Delta(\mathcal{L})$, its real part $\mathbf{P}_{\mathbb{R}}^* \setminus \Delta_{\mathbb{R}}(\mathcal{L})$ is in general disconnected and the topology of $\Sigma_{\mathbb{R}}^{(\lambda)}$ depends on the connected component λ belongs*

to. Define now

$$\widehat{E} = \{(\lambda, x) \mid \lambda \in \ell, x \in \Sigma^{(\lambda)}\} \subset \ell \times X.$$

Due to the transversality assumptions between ℓ and $\Delta(\mathcal{L})$ the variety \widehat{E} is smooth. We endow it with the symplectic structure $\widehat{\Omega}$ induced by $\omega_\ell \oplus \omega_X$.

The space \widehat{E} comes with two “projections”, $\pi : \widehat{E} \rightarrow \ell$ and $p_X : \widehat{E} \rightarrow X$, induced by the two projections from $\ell \times X$ to its factors. The first one is a Lefschetz fibration (whose base is $\ell \approx \mathbb{C}P^1$). The fact that the critical points of π are non-degenerate follows from the transversality assumptions on the intersection of ℓ and $\Delta(\mathcal{L})$. The second projection (which will not be used here) realizes \widehat{E} as the blow-up $\text{Bl}_B(X) \rightarrow X$ of X along the base locus B of the pencil ℓ (i.e. $B = \{x \in X \mid x \in \Sigma^{(\lambda)} \forall \lambda \in \ell\}$). The involutions c_H and c_X induce an anti-holomorphic involution on \widehat{E} which is also anti-symplectic with respect to $\widehat{\Omega}$.

Let $D \subset \ell$ be a closed disk which is invariant under c_H . Identify $\ell \setminus D$ with \mathbb{C} via an orientation preserving diffeomorphism which commutes with $(c_H, c_\mathbb{C})$, where $c_\mathbb{C}$ is the standard conjugation on \mathbb{C} . The real part $\ell_\mathbb{R} \setminus D$ of $\ell \setminus D$ is sent by this diffeomorphism to \mathbb{R} .

By restricting π to the complement of D we obtain a Lefschetz fibration $E = \pi^{-1}(\ell \setminus D)$ over $\ell \setminus D \cong \mathbb{C}$. We endow E with the symplectic structure Ω coming from $\widehat{\Omega}$ and by a slight abuse of notation denote its projection by $\pi : E \rightarrow \mathbb{C}$. Restricting the preceding anti-symplectic involution of \widehat{E} to E we obtain an anti-symplectic involution c_E on E which covers the standard conjugation $c_\mathbb{C}$ as in §6.5. The critical values of π are precisely $(\ell \setminus D) \cap \Delta(\mathcal{L})$. Some of them lie on $\ell_\mathbb{R}$ (i.e. the real axis) and the others come in pairs of conjugate points.

Note that $\ell_\mathbb{R} \setminus \Delta(\mathcal{L})$ might have several connected components. If $\lambda', \lambda'' \in \ell_\mathbb{R} \setminus \Delta(\mathcal{L})$ are in the same component then $\Sigma_\mathbb{R}^{(\lambda')}$ and $\Sigma_\mathbb{R}^{(\lambda')}$ are diffeomorphic, but otherwise not necessarily.

Consider now the fixed point locus $V = \text{Fix}(c_E) \subset E$. By Proposition 6.5.1, V is a Lagrangian cobordism. Its ends correspond to $\Sigma_\mathbb{R}^{(\lambda_-)}$ and $\Sigma_\mathbb{R}^{(\lambda_+)}$, where $\lambda_-, \lambda_+ \in \ell_\mathbb{R} \setminus D$ are close enough to the two boundary points of $\ell_\mathbb{R} \cap D$. As hinted above, any of the $\Sigma^{(\lambda_\pm)}$ might be disconnected. At the other extremity any of these ends might also be void.

Finally we address the issue of monotonicity. Assume that $\dim_\mathbb{C} X \geq 3$ and that the symplectic manifold $(\Sigma^{(\lambda)}, \omega_X|_{\Sigma^{(\lambda)}})$, $\lambda \notin \Delta(\mathcal{L})$, is monotone. By Proposition 6.5.1 the Lagrangian cobordism V is monotone.

Turning to more algebraic-geometric terms, here is a criterion that assures monotonicity of the $\Sigma^{(\lambda)}$'s. For an algebraic variety we denote by $-K_X$ its canonical class. The following follows easily from adjunction.

Proposition 6.5.2. *Let X be a Fano manifold with $\dim_\mathbb{C} X \geq 3$ and write $-K_X = rD$, with $r \in \mathbb{N}$ and D a divisor class. Further, suppose that $\mathcal{L} = qD$ with $0 < q \in \mathbb{Q}$ and $q < r$. Then the symplectic manifolds $(\Sigma^{(\lambda)}, \omega_X|_{\Sigma^{(\lambda)}})$, $\lambda \notin \Delta(\mathcal{L})$, are monotone. In particular V is a monotone Lagrangian cobordism.*

6.5.3. *A concrete example - real quadric surfaces.* We present here a concrete example of a real Lefschetz fibration associated to a pencil of complex quadric surfaces in \mathbb{CP}^3 . The example can be easily generalized to higher dimensions.

Let $X = \mathbb{CP}^3$ and $\mathcal{L} = \mathcal{O}_{\mathbb{CP}^3}(2)$, both endowed with their standard real structures (induced by complex conjugation). Clearly \mathcal{L} is very ample and gives rise to the so called degree-2 Veronese embedding which we describe shortly.

Using coordinates $[X_0 : X_1 : X_2 : X_3]$ on \mathbb{CP}^3 we identify the space $H^0(\mathcal{L})$ of sections of \mathcal{L} with the space of quadratic homogeneous polynomials $\lambda(\underline{X})$ in the variables $\underline{X} = (X_0, X_1, X_2, X_3)$:

$$(72) \quad \lambda(\underline{X}) = \sum_{0 \leq i \leq j \leq 3} a_{i,j} X_i X_j.$$

Taking $X_i X_j$, $0 \leq i \leq j \leq 3$, as a basis for this space we obtain an identifications $\mathbf{P} \cong \mathbb{CP}^9$ under which the projective embedding $X \hookrightarrow \mathbb{CP}^9$ is given by:

$$[z_0 : z_1 : z_2] \mapsto [z_0^2 : z_0 z_1 : \cdots : z_i z_j : \cdots : z_2 z_3 : z_3^2],$$

where the coordinates on the right-hand side go over all (i, j) with $0 \leq i \leq j \leq 3$.

The hyperplane section corresponding to the polynomial λ is a quadric surface

$$\Sigma^{(\lambda)} = \{[z_0 : z_1 : z_2 : z_3] \mid \lambda(z_0, z_1, z_3, z_3) = 0\} \subset \mathbb{CP}^3.$$

A straightforward calculation shows that $\lambda \in \Delta(\mathcal{L})$ if and only if

$$(73) \quad \det \begin{pmatrix} 2a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & 2a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & 2a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & 2a_{33} \end{pmatrix} = 0.$$

This shows that the discriminant $\Delta(\mathcal{L})$ is a variety of degree 4 in $\mathbf{P}^* \cong \mathbb{CP}^9$. The smooth stratum of $\Delta(\mathcal{L})$ consists of those λ 's where the matrix in (73) has rank 3.

The real part $\Delta_{\mathbb{R}}(\mathcal{L})$ of the discriminant consists of those polynomials λ which in addition to (73) have real coefficients (i.e. $a_{i,j} \in \mathbb{R}$ for every i, j).

It is well known that for $\lambda \notin \Delta(\mathcal{L})$ the variety $\Sigma^{(\lambda)}$ is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$, and moreover when viewed as a symplectic manifold (endowed with the structure induced from the projective embedding) it is symplectomorphic to $(\mathbb{CP}^1 \times \mathbb{CP}^1, 2\omega_{\mathbb{CP}^1} \oplus 2\omega_{\mathbb{CP}^1})$, where $\omega_{\mathbb{CP}^1}$ is normalized so that the area of \mathbb{CP}^1 is 1.

Consider now the following two sections

$$\lambda_0(\underline{X}) = X_0^2 + X_1^2 + X_2^2 - X_3^2, \quad \lambda_1(\underline{X}) = X_0 X_3 - X_1 X_2.$$

A simple calculation shows that $\lambda_0, \lambda_1 \notin \Delta(\mathcal{L})$. Denote the real part of $\Sigma^{(\lambda_i)}$ by $L^{(\lambda_i)}$, $i = 0, 1$. It is easy to see that $L^{(\lambda_1)}$ is a Lagrangian tours and moreover we can find a

symplectomorphism $\phi^{(\lambda_1)} : \Sigma^{(\lambda_1)} \rightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$ so that $\phi^{(\lambda_1)}(L^{(\lambda_1)})$ is the split torus $T = \mathbb{R}P^n \times \mathbb{R}P^1$. We fix such a diffeomorphism $\phi^{(\lambda_1)}$. Similarly, there is a symplectomorphism $\phi^{(\lambda_0)} : \Sigma^{(\lambda_0)} \rightarrow \mathbb{C}P^1 \times \mathbb{C}P^1$ that sends $L^{(\lambda_0)}$ to the Lagrangian sphere $S = \{(z, \bar{z}) \mid z \in \mathbb{C}P^1\} \subset \mathbb{C}P^1 \times \mathbb{C}P^1$ which is so-called the anti-diagonal.

We now consider the pencil $\ell \subset \mathbf{P}^*$ that passes through the two points λ_0 and λ_1 . Clearly ℓ is invariant under the anti-holomorphic involution c_H . We can parametrize ℓ by

$$\mathbb{C}P^1 \ni [t_0 : t_1] \mapsto \lambda_{[t_0:t_1]} := t_0\lambda_0 + t_1\lambda_1.$$

A simple calculation shows that the intersection points of ℓ with $\Delta(\mathcal{L})$ occur for the following values of $[t_0 : t_1]$:

$$(74) \quad [t_0 : t_1] \in \{[1 : 2], [1 : -2], [1 : 2i], [1 : -2i]\},$$

and that ℓ intersects $\Delta(\mathcal{L})$ only along the regular stratum. Moreover this intersection is transverse. See the left part of Figure 40.

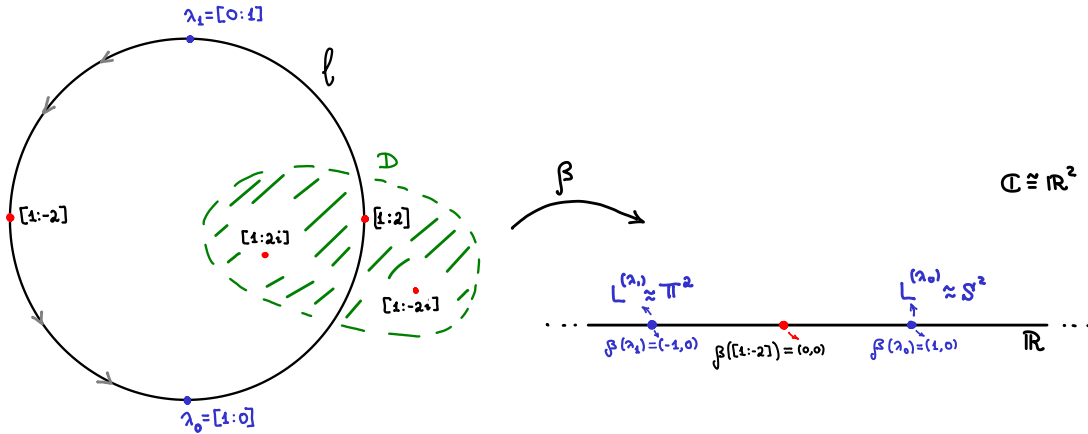


FIGURE 40. The real pencil ℓ on the left, and the image of $\ell \setminus D$ under β in \mathbb{C} .

We now appeal to the construction in §6.5.2. Below we will often identify $\mathbb{C} \cong \mathbb{R}^2$ in the obvious way. Choose a disk $D \subset \ell$ which is invariant under c_H and contains the point $[1 : 2], [1 : 2i], [1 : -2i]$ but not the point $[1 : -2]$. Fix an orientation preserving diffeomorphism $\beta : \ell \setminus D \rightarrow \mathbb{C} \cong \mathbb{R}^2$ such that:

$$\beta(\lambda_1) = (-1, 0), \quad \beta(\lambda_0) = (1, 0), \quad \beta([1 : -2]) = (0, 0).$$

See the right part of Figure 40. From now on we use the identification β implicitly and simply write $\lambda_1 = (-1, 0)$, $\lambda_0 = (1, 0)$.

Restricting \hat{E} to $\ell \setminus D$ and applying a base change via β we obtain a Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$ with general fiber $\mathbb{C}P^1 \times \mathbb{C}P^1$ and with a real structure. Since the minimal Chern

number of the general fiber is $c_1^{\min} = 2$, E is a strongly monotone Lefschetz fibration in the sense of Definition 3.2.1. Its monotonicity class is $*$ = (0).

The projection π has exactly one critical value at $0 \in \mathbb{C}$ (corresponding to $[1 : -2] \in \ell$). The real part V of E is a cobordism with one negative end associated to $L^- = L^{(\lambda_1)}$ which is a Lagrangian torus, and one positive end associated to $L^+ = L^{(\lambda_0)}$ which is a Lagrangian sphere. By Proposition 6.5.2 V is monotone and a simple calculation shows that it has minimal Maslov number $N_V = 2$. Interestingly we have $N_{L^-} = 2$ while $N_{L^+} = 4$. Note also that $d_{L^-} = d_{L^+} = 0$, hence V is of the right monotonicity class $*$ = (0).

Transforming V to a negative ended cobordism. In order to obtain a cobordism with only negative ends (as considered in the rest of the paper) we proceed as follows. Take the Lefschetz fibration $\pi : E \rightarrow \mathbb{C}$ and $V \subset E$ as constructed above. Recall that $0 \in \mathbb{C}$ was the (single) critical value of π . Consider a smooth embedding $\alpha' : [0, \infty) \rightarrow \mathbb{R}^2$ so that:

- (1) $\alpha'(t) = (t, 0)$ for every $0 \leq t \leq 1$.
- (2) For $1 < t$, α' lies in the lower half plane and $\alpha'(2) = (0, -1)$.
- (3) For every $2 \leq t$, $\alpha'(t) = (2 - t, -1)$.

Now take the part of the cobordism V that lies over $(-\infty, 1] \times \mathbb{R} \subset \mathbb{R}^2$ and glue to its right hand side the trail of the Lagrangian sphere $L^{(\lambda_0)} = V|_{(1,0)}$ along the curve $\alpha'|_{[1,\infty)}$. Denote the result by W . It is easy to see that W is a smooth Lagrangian cobordism with two negative ends. The lower end is a Lagrangian sphere and the upper end is a Lagrangian torus, both living inside symplectic manifolds that are symplectomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$. See Figure 41.

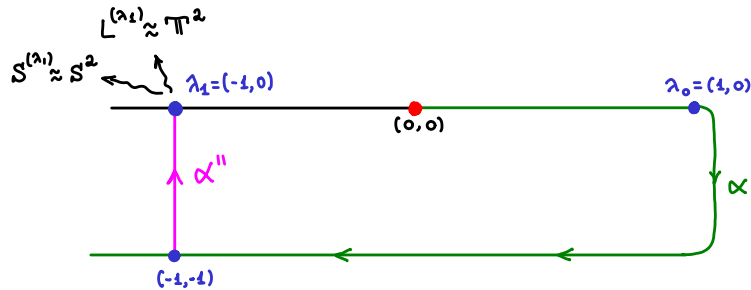


FIGURE 41. The cobordism W with two negative ends, and the parallel transport of the sphere $L^{(\lambda_0)}$ to the fiber over λ_1 .

Note that the Lefschetz fibration E is not tame. Therefore In order to apply the cone decomposition from Corollary 5.1.1 we need to identify fibers over different ends. To this end, denote by α'' the straight segment connecting $\alpha'(3) = (-1, -1)$ to $\lambda_1 = (-1, 0)$. Denote by $\alpha = \alpha'|_{[1,3]} * \alpha''$ the concatenation of $\alpha'|_{[1,3]}$ with α'' . Denote by $\Pi_\alpha : E_{\lambda_0} \rightarrow E_{\lambda_1}$ the parallel transport along α . Let $S^{(\lambda_1)} = \Pi_\alpha(L^{(\lambda_0)})$ be the parallel transport of the Lagrangian sphere

$L^{(\lambda_0)}$ to the fiber $\Sigma^{(\lambda_1)} = E_{\lambda_1}$ of E over λ_1 . See Figure 41. By Corollary 5.1.1 we have in $D\mathcal{Fuk}^*(\Sigma^{(\lambda_1)})$ an isomorphism:

$$(75) \quad S^{(\lambda_1)} \cong \text{cone}(S_1 \otimes E \longrightarrow L^{(\lambda_1)}),$$

where $S_1 \subset \Sigma^{(\lambda_1)}$ is the vanishing cycle associated to the critical point of π over 0 and the path $\alpha'|_{[0,3]} * \alpha''$. According to (57), the space E is $HF(\hat{S}_1, W)$, where \hat{S}_1 is the matching cycle emanating from z_1 , which lies in a suitable extension of the fibration E (see §4.4.2).

In our case, it is not hard to see that \hat{S}_1 intersects W at a single point and the intersection is transverse. Therefore E is a 1-dimensional space. Applying $\phi^{(\lambda_1)}$ to (75) we now obtain the following isomorphism in $D\mathcal{Fuk}^*(\mathbb{CP}^1 \times \mathbb{CP}^1)$:

$$\phi^{(\lambda_1)}(S^{(\lambda_1)}) \cong \text{cone}(\phi^{(\lambda_1)}(S_1) \longrightarrow T).$$

By a result of Hind [Hin] all Lagrangian spheres in $\mathbb{CP}^1 \times \mathbb{CP}^1$ are Hamiltonian isotopic. In particular $\phi^{(\lambda_1)}(S^{(\lambda_1)})$ and $\phi^{(\lambda_1)}(S_1)$ are both Hamiltonian isotopic to the anti-diagonal S . It follows that:

$$(76) \quad S \cong \text{cone}(S \longrightarrow T).$$

By rotating the exact triangle corresponding to (76) we obtain the following result:

Corollary 6.5.3. *Let $M = \mathbb{CP}^1 \times \mathbb{CP}^1$, endowed with the symplectic structure $\omega_{\mathbb{CP}^1} \oplus \omega_{\mathbb{CP}^1}$. Denote by $S = \{(z, \bar{z}) \mid z \in \mathbb{CP}^1\} \subset M$ the anti-diagonal and by $T = \mathbb{RP}^1 \times \mathbb{RP}^1 \subset M$ the split torus. Then in $D\mathcal{Fuk}^*(M)$ there is an isomorphism*

$$(77) \quad T \cong \text{cone}(S \longrightarrow S).$$

- Remarks.*
- a. The existence of an isomorphism of the type (77) could probably be derived also by the following construction whose details need to be precisely worked out. Consider a Hamiltonian isotopic copy S' of S so that S' intersects S transversely at exactly two points. By performing Lagrangian surgery of S' and S at the intersection points (with appropriate choices of handles) one obtains a Lagrangian torus $T' \subset M$. Moreover, for a suitable choice of S' and choices of handles the torus T' should be Hamiltonian isotopic to the split torus T . Applying the “figure-Y” surgery construction from [BC2] we obtain a cobordism V in $\mathbb{R}^2 \times M$ with two negative ends S, S' and one positive end T' . The cobordism V should also be monotone for suitable choices of handles in the figure-Y surgery. The cone decomposition in (77) would now follow from the main results of [BC3].
 - b. Our work does not provide much information about the precise morphism $S \longrightarrow S$ from (77). It would be interesting to determine the precise map and also to figure out how (77) behaves with respect to grading (in this case a \mathbb{Z}_2 -grading).

A few variations on the same example. One can alter the construction of E and V to obtain a Lefschetz fibrations $\pi : E' \rightarrow \mathbb{C}$ with more critical values. This can be done for example by choosing the disk D to contain the point $[1 : -2]$ and none of the other points from (74). The result will then be a fibration with three critical values - one lying on the x -axis and another pair of critical points conjugate one to the other. The cobordism V in this case would still be between a Lagrangian sphere and a torus.

If one chooses the disk D not to contain any of the points in (74) and its center to lie somewhere along the interval $[1 : x]$, $x \in [-2, 2]$, then the fibration will have four critical values, two real ones and two conjugate ones. The cobordism V will have a Lagrangian S^2 on its both ends, and the topology of V will still be non-trivial (i.e. V will not be diffeomorphic to $\mathbb{R} \times S^2$). A similar example with Lagrangian \mathbb{T}^2 's on both ends can be constructed by taking the disk to have its center somewhere along $[1 : x]$, $x > 2$.

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